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ON THE PROPERTIES OF A GAS OF CHARGED PARTICLES

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Introduction.

The behaviour of systems having numerous degrees of freedom and endowed with electric charges presents problems of high complexity. Although many of the properties of systems of this kind may be explained within classical physics, a more consistent description was obtained only after the introduction of quantum mechanics.

An example studied with particular care is the so-called electron gas. We shall not attempt here a historical survey of the viewpoints and mathematical treatments of electron gases. It may only be recalled that a striking simplification was obtained when SOMMERFELD introduced the idea of a gas of free electrons, subject only to the exclusion principle, but uninfluenced by the mutual electric forces. In this way, an approximate account was given of many of the properties characteristic of metallic electrons. One of the major results was the smallness of the thermal energy of the gas at usual temperatures. The model could be refined so as to allow for periodic static potentials, which led to a separation of the electronic states into energy bands and to an explanation of essential properties of the different types of solids. At the same time the free electron picture, as applied in the Thomas-Fermi model, could account for properties of atoms in an averaged manner.

It was thus apparent that the Sommerfeld theory in several respects closely represented the properties of, e.g., valence electrons in metals. Still, it seemed difficult to give a direct justification of the model on the basis of more rigorous treatments, because the interaction appeared to have a dominating influence on the behaviour of the electrons. It was not easy to foresee the limitations of the theory, and in problems where it was valid in the first approximation ambiguities arose in the more detailed

treatments. This was the more unsatisfactory since some phenomena, as superconductivity, apparently defied any explanation within the simple theory of metals.

An interesting attempt to describe the dynamics of an electron gas in a systematic way, with inclusion of the interaction between the particles, was made by BLOCH (1933, 1934) on the basis of the Thomas-Fermi model. Although in the treatment further simplifications—introduction of hydrodynamic concepts—were made, it was possible to arrive at a number of simple results as regards the modes of excitation of the system. More recently, TOMONAGA (1950) presented a thorough discussion of the behaviour of a one-dimensional gas, again for fields varying slowly in space. Further, BOHM and collaborators (1949—1953) have published extensive studies of many problems in the three-dimensional case (see especially BOHM and PINES, 1953). The treatments by BLOCH, TOMONAGA, and BOHM indicate the great modification suffered by the system of originally free particles when the interaction is introduced.

This great change in the properties is—as empirically evident—mainly a change in the electromagnetic properties of the system. As in the treatment by BOHM, the interest therefore in the first instance centers on the behaviour of the electromagnetic field. However, we shall here make the further step of writing the equations of motion of the total system as equations for the electromagnetic field only. One would expect this to be possible, since the behaviour of the particles is revealed only through the field to which they give rise. The motion of a particle is then no longer determined by the interaction with numerous other particles, but only by the field. This elimination of the particles in the description of the properties of a medium is nothing but the idea at the basis of Maxwell's field equations in matter. The same idea is found in the static descriptions of THOMAS and FERMI, and of HARTREE. From the present point of view it is less adequate to introduce elaborations of these models of the kind contained in determinant wave functions, or in the description by correlations, where one tries to include energies of order e^2, or higher, by calculating the effect of interaction between two or more particles. The difficulties in a correlation description was evidenced in calculations by HEISENBERG (1947), where

divergencies appeared for long wavelengths. Such kinds of approach become more consistent if one introduces a screened Coulomb interaction between the particles (PINES, 1953; LANDSBERG, 1949; WOHLFARTH, 1950).

We should now make a more definite statement as to the field entering in our description: By the classical field in the substance we mean always the total field acting on and eventually due to classical, or external, charges. We wish to derive the corresponding field equations, which turn out to be of the type of Maxwell's field equations in matter. This attitude might seem to require a detailed justification because fluctuations could destroy the uniqueness of the description. However, the classical field equations define, on the contrary, what one understands by a large system of interacting particles. Thus, it is important to realize that all fluctuations occurring are contained in the present description, simply because it gives the classical equations of motion of the system. A quantization of the equations may be performed by applying the methods familiar from field theory. With such methods it is possible to derive the fluctuations in the system, on similar lines as followed by BLOCH (1934) in his treatment of the Thomas-Fermi gas. In the following, we shall try to amplify these brief remarks, both as regards which is the field considered and as to the uniqueness of the field equations.

Accepting the above, an immediate task is to find what simplifications and modifications can result when the dynamical properties are contained in field equations for the electromagnetic field. Now, even though it can be difficult to derive more exact field equations on explicit form, one may be able to find easily the type of the equations and to give approximate estimates in various limiting cases. One can then, with confidence, attempt an approach to the field equations both from a theoretical and an empirical point of view, and consider the general consequences of the structure of the equations. Of course, at the same time as one has an equation for the electromagnetic field, an equation for the particle field will be obtained (cf. Appendix). The latter equation is of particular interest, for instance, in scattering phenomena. For the present, we are concerned mainly with the electromagnetic field, but one has the choice of developing with higher accuracy the description of either of the two fields.

It will be useful, then, to discuss general field equations of classical type. We limit the treatment to linear field equations, and define these by the dielectric operators of the field. Linear field equations will seem to cover a wide class of phenomena, but in some cases the linear approximation is not sufficient. Terms of higher order appear in, e.g., the Thomas-Fermi atomic model and phenomena as those studied by ALFVÉN (1950) and his school.

In the derivations, in the present paper, of classical linear field equations for a gas of charged particles several further approximations are made. We compute at first the dielectric constants to first order in e^2, starting from the picture of a free gas of particles. One gets then a number of well-known results regarding the properties of free gases, but finds essential improvements in some cases. When the computations are carried out to higher order, one encounters further corrections to the familiar descriptions.* The first order equations contain the main features of the linear field equations of the system. In some cases they can represent the exact solution and yet contain a by no means small polarization (cf. § 5, p. 48). From such first order treatment one sees perhaps best that the field equation concerns the total classical field, since the particle motions—and thus the polarization—are determined only by this field.

As regards the question of the justification of the approximation method applied in the following, where one starts from a free gas and computes the dielectric constant only to first order in e^2, it would seem most appropriate to calculate the linear field equations to a high, or infinite, order and make a comparison with the first order treatment. Employing the powerful technique of quantum field theory, one can indeed get expressions for the linear electromagnetic and particle field equations. We use a more modest approach, and discuss the behaviour of one electron moving in the field calculated to second order. The electron then

* By a computation to higher order we understand a procedure of iteration applied only in deriving the field equations; when these are obtained we do not attempt approximations at a later stage.

In a sense the accuracy of the calculation to first order is dependent on e^2 being small. We can in fact form one dimensionless quantity from e^2 and the density of the gas, ϱ. In the following we use the quantity $\chi^2 = \left[me^2/\hbar^2 (3\pi^5\varrho)^{\frac{1}{3}}\right]$. The value of χ^2 must determine the properties of the gas. The free particle picture is valid if χ^2 is small compared to 1, as in dense gases.

interacts with itself, and we consider in some detail the effect of the damping of motion and of the self-energy. As to the particular example of a one-dimensional gas the present perturbation method, applicable to all wavelengths of the field, gives in the region of long wavelengths precisely the equations deduced by TOMONAGA (1950).

We have not yet commented on the crucial question of the connection with statistical mechanics. In spite of the circumstance that we are concerned, basically, with a conservative system, the field turns out to be of non-conservative type and we encounter dissipation effects of peculiar kind. Thus, absorption of energy and momentum occurs via processes not unlike those resulting from viscosity in a liquid. The absorption effect appears as a finite imaginary operator in the equations for the retarded field. It may be said to derive from an extreme in entropy increase and trend towards statistical equilibrium, and must result in a small thermal excitation of the system. Just the former circumstance implies an unambiguous classical state —and behaviour in time—of the system, where mechanical considerations alone could not suffice. Incidentally, we have therefore employed the word 'classical' in the simple sense: with neglect of statistical or quantum mechanical fluctuations. The statistical fluctuations about the average absorption can be obtained in a straightforward manner from correspondence arguments.

In the following only some of the above general questions are treated in detail. First, in § 1, the field equations in matter are introduced. The equations holding for a nearly free gas are computed classically in § 2, and by quantum mechanical treatment of the particles in § 3. In the static limit these calculations are in line with the method introduced by O. KLEIN, whose point of view was similar to the present one (KLEIN, 1945; LINDHARD, 1946). By way of illustration we consider in § 4 the application to stopping problems, and self-energy, for charged particles passing through matter (cf. also Appendix). The improvement of the free electron picture is discussed in § 5. In a subsequent paper* will be treated the connection between the thermal properties and the field equations, and also the absorp-

* To appear in Dan. Mat. Fys. Medd., and in the following referred to as II.

tion and reflection of a transverse wave by a metallic surface (cf. moreover LINDHARD, 1953).

Before entering on the more formal treatment we may add a few words as to the general formulation given here, and concerning the results of a discussion of the free electron gas. It might perhaps seem that the free gas is a quite specialized example, but it should be appreciated, first, that its applicability to atomic systems containing many electrons is quite wide. The proper methods for a treatment of many-particle systems should just derive from an understanding of the simple example of a free gas.

Second, the electron gas is a system in which the forces between the particles are known. If it is attempted to discuss systems where the forces are less well understood, as in an assembly of nucleons, a detailed description of the simpler case may be of considerable help. It is of special interest that a system of interacting particles exhibits several typical properties, for which the character of the forces is usually not decisive.

One may here particularly mention the connection between the independent-particle model and the collective model of atomic nuclei. These two cases we meet already for electronic systems, which in numerous respects show an extraordinary likeness to a nucleus. The collective and independent-particle features are intimately connected in the present field equations, and though the total effect of the particles, as expressed by the field polarization, can be very large, yet each particle may move nearly as if it were free (in the more general case, the particles have no longer free particle equations, but remain independent). A considerable simplification results since the two descriptions do not appear as entirely different cases, but may be treated as one.

§ 1. Field equations in matter.

In this paragraph are described the more general features of electromagnetic field equations in matter, as a preliminary to the calculations in the following. One of the questions is, simply, the introduction of a suitable notation and terminology; we choose to formulate the field equations by means of the dielectric con-

stants of the field. Actual expressions for dielectric constants are to be found in § 2 and § 3. Further, the discussion here is meant to illustrate characteristic phenomena, as energy dissipation, that are contained in the field equations.

It is desirable to use a description similar to that of the Maxwell equations in matter. In these equations, the properties of a substance are determined by the values of the dielectric constant ε, the permeability μ, and the conductivity σ. Expanding the field in harmonic components in time, the quantities ε and μ may be considered as functions of the frequency ω; the conductivity σ can be included in the dielectric constant and determines then the imaginary part of ε. One distinguishes, on the one hand, between the general equations which do not contain the properties of the medium,

$$\left.\begin{aligned} &\operatorname{div} \vec{D} = 4\pi\varrho_0, \quad \operatorname{rot} \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \\ &\operatorname{div} \vec{B} = 0 \quad , \quad \operatorname{rot} \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c}\vec{j}_0, \end{aligned}\right\} \quad (1.1)$$

and, on the other hand, the more special equations where the electromagnetic properties enter directly,

$$\vec{B}(\vec{r}, \omega) = \mu(\omega) \cdot \vec{H}(\vec{r}, \omega), \quad \vec{D}(\vec{r}, \omega) = \varepsilon(\omega) \cdot \vec{E}(\vec{r}, \omega), \quad (1.2)$$

$B(\vec{r}, \omega)$, etc., being field components with time dependence $\exp(-i\omega t)$. While the equations (1.1) are quite general, the assumption (1.2) leads directly to linear field equations.

Using a description of the kind (1.2), one is able to give a satisfactory account of many properties of matter. Still, for a number of phenomena not only the time variation of the field, as contained in (1.2), but even its variation in space will have a decisive influence on the values of ε and μ. Moreover, the general interpretation of (1.2) and the transcription of the Maxwell equations into equations for the electromagnetic potentials remain ambiguous if one assumes that ε and μ are functions only of ω. We shall find, however, that when (1.2) is generalized so as to take into account the spatial variation of the fields, such ambiguities disappear.

We may now discuss the general form of linear field equations

in matter, without immediately trying to find the connection with the Maxwell equations (1.1) and (1.2). Let us consider the microscopic behaviour of an electromagnetic field in an extended medium. We may for the present regard the medium as infinite, isotropic, and homogeneous, the latter even over minute regions in space, and with properties independent of time. As will be discussed below, we shall be concerned with the classical, or average, field and at first look apart from fluctuations. In the case under consideration, it will be convenient to describe the fields by their Fourier components in space and time. Indeed, due to the invariance towards displacements in the medium, each Fourier component of the field must be proportional to the same Fourier component of the sources. The factor of proportionality will be some function depending both on the wave vector and frequency of the component in question. Considering, in the first instance, the transverse (divergence-free) part of the field we may express this circumstance by introducing a function which we shall call the transverse dielectric constant. The reasons for this particular formulation and the connection with the Maxwell equations in matter will appear presently. We thus get for the transverse part of the vector potential

$$\left(k^2 - \frac{\omega^2}{c^2}\,\varepsilon^{tr}\right)\vec{A}^{tr}(\vec{k},\omega) = \frac{4\pi}{c}\,\vec{j}_0^{\,tr}(\vec{k},\omega), \tag{1.3}$$

where $\varepsilon^{tr} = \varepsilon^{tr}(\vec{k},\omega)$ is a function of the wave vector k and the frequency ω. The connection between $\vec{A}(\vec{r},t)$ and $\vec{A}(\vec{k},\omega)$ is given by $\vec{A}(\vec{r},t) = \sum_{k,\omega} \vec{A}(\vec{k},\omega)\cdot\exp(i\vec{k}\cdot\vec{r} - i\omega t)$, and in the same way $\vec{j}_0^{\,tr}(\vec{k},\omega)$ is a Fourier component of the transverse current of the sources of the field. The transverse character of the field is expressed by $\vec{k}\cdot\vec{A}^{tr}(\vec{k},\omega) = 0$, so that $\vec{A}^{tr}(\vec{k},\omega) = \vec{A}(\vec{k},\omega) - \vec{k}(\vec{k}\cdot\vec{A}(\vec{k},\omega))/k^2$, and similarly for $\vec{j}_0^{\,tr}(\vec{k},\omega)$. On account of the explicit appearance of the sources equation (1.3) describes a case of forced vibrations of the medium; the solution of the homogeneous equation is discussed more closely in II.

We note that, since $\vec{A}$ and $\vec{j}_0$ are real functions in space and time, their Fourier components $(-\vec{k},-\omega)$ must be complex con-

jugate of the components $(\vec{k}, \omega)$, which implies that $\varepsilon^{tr}(-\vec{k}, -\omega) = \varepsilon^{tr*}(\vec{k}, \omega)$. The dielectric constants in the following will generally be of the type $\varepsilon^{tr}(\vec{k}, -\omega) = \varepsilon^{tr*}(\vec{k}, \omega)$, as is characteristic for retarded and advanced field equations. We shall always be concerned with the retarded dielectric constants, i. e. retarded field equations, except when expressly otherwise stated.

For the electric potential we can choose to write, in a similar way, introducing a new function which we call the longitudinal dielectric constant,

$$\varepsilon^{l} \cdot k^2 \, \Phi(\vec{k}, \omega) = 4\pi \, \varrho_0(\vec{k}, \omega), \tag{1.4}$$

where the notation is as in (1.3) and $\varepsilon^{l} = \varepsilon^{l}(k, \omega)$ depends on the two variables k and ω. The quantity $\varrho_0(\vec{k}, \omega)$ on the right is the Fourier component of the source density of electric charge. When the description (1.4) is used, it is assumed that the gauge is chosen such that the longitudinal vector potential vanishes. It may perhaps already here be stressed that, for fields in material substances, one has directly given a natural system of reference and it is often not convenient to write the field equations in an invariant manner.

The fields derived from the above potentials are

$$\vec{E} = -\operatorname{grad} \Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \operatorname{rot} \vec{A} \tag{1.5}$$

and they are the true classical fields acting at a space-time point $(\vec{r}, t)$ in the medium. We consider the retarded ε's so that the fields in (1.5) are the retarded fields. The letters E and B are chosen for these quantities because they can be interpreted as the electric field and the magnetic induction of the Maxwell equations, as we shall now show. We have accounted for the properties of the medium by the two functions ε^{tr} and ε^{l}, instead of ε and μ in the equations (1.2). The connection between the two formulations is apparent when a function $\mu(k, \omega)$ is defined by

$$k^2 \left(1 - \frac{1}{\mu(k, \omega)}\right) = \frac{\omega^2}{c^2} \left(\varepsilon^{tr}(k, \omega) - \varepsilon^{l}(k, \omega)\right), \tag{1.6}$$

for then equation (1.3) becomes, introducing a dielectric constant ε, equal to the longitudinal one, ε^l,

$$\left(\frac{k^2}{\mu}-\frac{\omega^2}{c^2}\,\varepsilon\right)\vec{A}^{tr}(\vec{k},\omega)=\frac{4\pi}{c}\vec{j}_0^{\,tr}(\vec{k},\omega). \tag{1.7}$$

With this notation one can introduce a magnetic field, H, and an electric displacement, D, from the equations

$$\left.\begin{aligned}\vec{H}(\vec{k},\omega)&=\frac{1}{\mu(k,\omega)}\cdot\vec{B}(\vec{k},\omega)\\ \vec{D}(\vec{k},\omega)&=\varepsilon(k,\omega)\cdot\vec{E}(\vec{k},\omega).\end{aligned}\right\} \tag{1.8}$$

One then obtains the Maxwell equations (1.1) in space and time for the quantities $\vec{B}(\vec{r},t)$, $\vec{H}(\vec{r},t)$, $\vec{E}(\vec{r},t)$ and $\vec{D}(\vec{r},t)$. This formulation of the field equations, where μ, H and D are introduced, is often convenient. But in the following we shall usually describe the behaviour of the medium by using only ε^{tr} and ε^l, solving the equations (1.3) and (1.4) with respect to the potentials. In this connection it may also be emphasized that μ, according to (1.6), is determined unambiguously only when ε^l is known. Thus, if one treats exclusively the transverse field it can be convenient to interpret the Maxwell equations (1.8) in the manner that the permeability is 1, or $H = B$, and the dielectric constant is the transverse dielectric constant.

The permeability and the dielectric constants defined above should be considered as complex functions; their imaginary parts are closely connected with the energy absorption by the medium. We note in particular that the connection between the Fourier components of the induced transverse and longitudinal current densities and total electric fields are given by

$$\vec{j}_{\text{ind}}^{\,\{tr}_{l}(\vec{k},\omega)=(\vec{j}-\vec{j}_0)^{\{tr}_{l}=\frac{i\omega}{4\pi}\left(\varepsilon^{\{tr}_{l}(k,\omega)-1\right)\vec{E}^{\{tr}_{l}(\vec{k},\omega), \tag{1.9}$$

the equation holding, as indicated, for the transverse and longitudinal fields separately. Equation (1.9) leads us to define transverse and longitudinal conductivities as, respectively,

$$\sigma^{\mu\nu}(k,\omega) = \frac{\omega}{4\pi} \operatorname{Im}\left(\varepsilon^{\mu\nu}(k,\omega) - 1\right), \tag{1.10}$$

where $\operatorname{Im}(x)$ denotes the imaginary part of x. The real parts of the equations (1.10) correspond to the polarizabilities of the medium.

We now turn to considerations of a more general kind, based only on the Maxwell equations (1.1); we do this primarily in order to give an account of the phenomenon of absorption by the medium. The absorption does not express a lack of conservation of energy of the total system. Indeed, we shall derive it in § 2 and § 3 from conservative equations of motion. The absorption means only that a distribution takes place over a large number of degrees of freedom, and the corresponding motion cannot be regained as an ordered motion. In a more detailed description, the absorption by the medium can be described by thermodynamical and statistical mechanical parameters, and observed as, e. g., a rise in temperature of the system.

The conservation equations for energy and momentum are derived in the familiar manner from the Maxwell equations (1.1). We shall write them down in full so as to show their contents and interpretation in descriptions of the present kind. We find from (1.1) the energy equation

$$\left.\begin{aligned} &-\frac{1}{8\pi}\cdot\frac{\partial}{\partial t}(\vec{D}\cdot\vec{E}+\vec{H}\cdot\vec{B}) - \frac{c}{4\pi}\operatorname{div}(\vec{E}\times\vec{H}) \\ &= \frac{1}{8\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} - \vec{D}\cdot\frac{\partial\vec{E}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{B}\cdot\frac{\partial\vec{H}}{\partial t}\right) + \vec{E}\cdot\vec{j}_0, \end{aligned}\right\} \tag{1.11}$$

where the first and second terms on the left are minus the time derivative of the energy density and minus the divergence of the energy current, respectively. The last term on the right is the work done by the field on the sources. The bracketed term on the right is of a peculiar kind; it is often assumed to vanish in homogeneous media. However, applying (1.4), (1.7), and (1.8), we find—by integration over a space-time interval—that it represents the work done by the field on the medium, and it gives the rate dW/dt at which energy dissipates from the field in an

irreversible manner. We may express this result in terms of the potentials, and find directly from (1.8), introducing the conductivities from (1.10),

$$\Delta W = \sum_{k,\omega} \left(\frac{\omega^2}{c^2} \sigma^{tr}(k,\omega) \left| \vec{A}^{tr}(\vec{k},\omega) \right|^2 + k^2 \sigma^{l}(k,\omega) \left| \Phi(\vec{k},\omega) \right|^2 \right) \Delta t, \quad (1.12)$$

for the absorption in the time interval Δt. To be more precise, eq. (1.12) includes the dissipation of energy, if any, through the boundaries of the medium.

As to the equations for the momentum of the field, we apply again (1.1) and get for the component in the direction of the x-axis

$$\left.\begin{aligned} &-\frac{1}{4\pi c}\frac{\partial}{\partial t}(\vec{D}\times\vec{B})_x + \operatorname{div}_i T_{xi} \\ &= \frac{1}{8\pi}\left(\vec{D}\cdot\frac{\partial \vec{E}}{\partial x} - \vec{E}\cdot\frac{\partial \vec{D}}{\partial x} + \vec{B}\cdot\frac{\partial \vec{H}}{\partial x} - \vec{H}\cdot\frac{\partial \vec{B}}{\partial x}\right) \\ &\quad + \varrho_0 \cdot E_x + \frac{1}{c}\left(\vec{j}_0 \times \vec{B}\right)_x, \end{aligned}\right\} \quad (1.13)$$

where

$$T_{si} = \frac{1}{4\pi}\left(E_s D_i + H_s B_i - \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\cdot\delta_{si}\right)$$

is Maxwell's stress tensor in matter. From equation (1.13) we find that the momentum density is $(1/4\pi c)\cdot(\vec{D}\times\vec{B})$, which leads to the non-symmetric energy-momentum tensor of MINKOWSKI.* We shall in the discussion of reflection and transmission of light waves by metallic surfaces, in II, attempt to show the direct significance of this property of the energy-momentum tensor. It must be remembered that, if one will ascribe values to quantities as energy and momentum, one shall not primarily look for cases where they remain unchanged, but consider processes as absorption where exchanges of these quantities occur. As to the right-hand side of equation (1.13), the last term is the Lorentz

* Cf. C. MOLLER, The Theory of Relativity, Oxford 1952, p. 203. The Minkowski tensor has been adopted, too, in the 1951-edition of Die Relativitätstheorie by M.v. LAUE.

force density on the sources, while the first term corresponds to the first term on the right of (1.11), and is the rate at which the momentum density is taken up by the medium. The average rate at which momentum density is absorbed can be obtained from (1.12) when multiplying by the factor $\vec{k}/\omega$ inside the brackets.

We are thus able to account for the momentum and energy exchanged between the sources, the field, and the medium. In the case where the sources are outside the medium and the field is determined by boundary conditions, the energy dissipation can be found from (1.11) or (1.12), and the momentum absorption from the corresponding expressions. It may here be noted that, if the sources are inside the medium and not appreciably affected by the above momentum and energy exchange, a detailed treatment of the phenomena as a rule will be much simpler than in boundary problems or cases where the reaction of the sources must be included in the description.

So far the absorption was considered to take place in a continuous manner. When attempting to subdivide the absorption by the medium into elementary processes we must take into account that the quantum of energy corresponding to a frequency ω is $\hbar\omega$. It may be concluded immediately from (1.12) that the probability for the absorption of a quantum of frequency ω and wave vector k is, during a time interval Δt,

$$p(\vec{k},\omega) = \Delta t \cdot \left(\frac{\omega}{\hbar c} \sigma^{tr}(k,\omega) \left| \vec{A}^{tr}(\vec{k},\omega) \right|^2 + \frac{k^2}{\hbar\omega} \sigma^{l}(k,\omega) \left| \Phi(\vec{k},\omega) \right|^2 \right). \quad (1.14)$$

This expression represents a statistical mechanical probability, and in contrast to the previous equations it depends directly on quantum theory, through the explicit appearance of the constant h. From (1.14) can be derived in particular the fluctuations in energy and momentum absorption. We note further that, while the probability for absorption $\Delta p(\vec{k}, \omega)$ for quanta $\hbar\omega$ is positive for positive ω, it becomes negative for negative ω, corresponding to a transfer of negative energy to the field. It is a matter of convention whether one will introduce instead only the numerical value of ω, and accordingly only positive energies. In § 4 will

be discussed a few interesting and simple consequences of equation (1.14).

In the description used here, represented by for instance (1.8) together with (1.1), the quantities ε and $1/\mu$ were considered as numbers in k, ω-space and correspondingly as operators in ordinary space. This formulation will be sufficient for the present purposes. But it is seen that ε and $1/\mu$ in (1.8) may be more general operators, and not necessarily linear. The other formulae in this paragraph can then be similarly interpreted.

The question of measuring the field quantities, like E, D, etc., is a rather subtle one. The dependence of the dielectric constants on wavelength and frequency will evidently play an important role for such measurements. Actual examples of this dependence can be found in the following sections. Even though the description is best illustrated by such examples we shall make here a few remarks on the measuring problem in general. The familiar rule for measurements of the fields is that, for instance, E and D are the fields in crevasses in the medium, cut parallel and perpendicular to the field itself. But let us take into account that at the surface of the medium the fields do not change abruptly (as they would if the ratio between D and E were constant in the medium) and suppose that they vary in some smooth manner over a distance, d, from the surface. If now we cut crevasses of the above kind, though with dimensions smaller than or of the order of d, the fields arising in these crevasses are no longer E and D, because the polarization has changed character. In fact, for sufficiently small crevasses, the measurements must all give E, because, for high values of k, the dielectric constant tends to 1 so that the polarization disappears. This corresponds to the circumstance that the field acting on a classical point charge is E. Exceptions from the mentioned rule for measuring E and D will occur in many other cases; the rule is disobeyed if only short time intervals are allowed for the measurement, and there are cases too where the dielectric constants even for long wavelengths depend strongly on the wavelength of the field.

§ 2. Semi-classical treatment of an electron gas.

We shall first contemplate the behaviour of an electron gas in classical mechanics. The properties of the gas will be described in terms of the dielectric constants of the electromagnetic field equations introduced in § 1. In the first part of the present paragraph, a more formal derivation of the field equations is given; in the second part, a comparison is made with the results of other authors. The question of the justification of the calculation will be taken up in § 5.

The calculations are based on the previously mentioned picture of the dynamics of a system of electrons: There exists a total field in space and time, $\Phi(\vec{r}, t)$, $\vec{A}(\vec{r}, t)$, in which the separate electrons move. This motion gives rise to an induced charge and current density, ϱ_{ind} and $\vec{j}_{\text{ind}}$, which quantities will be functions of the total field. Besides the induced densities there may be source densities, $\varrho_0(\vec{r}, t)$ and $\vec{j}_0(\vec{r}, t)$, distributed within the system. The latter densities can, for instance, correspond to a charged particle passing through the gas. From MAXWELL's equations for empty space, with charge density $\varrho_{\text{ind}} + \varrho_0$ and current density $\vec{j}_{\text{ind}} + \vec{j}_0$, one then derives the field equations for the total field, of the type (1.3) and (1.4). In these equations the source densities are ϱ_0 and $\vec{j}_0$. We shall suppose that the electronic motion differs only slightly from that in the undisturbed state of the system. This implies, generally, that the field equations will become linear.

Quite apart from the question of the applicability of a linear treatment, and of classical theory, the particular approximation of the first order picture is that the motion of the individual particles in the gas gives rise to and derives from a field determined by MAXWELL's equations for empty space. But it will be clear that one should rather assume that the field connected with the motion of the individual electron is governed by the final field equations. Such a treatment on more precise lines is somewhat more complicated and we shall show that it is not always needed. As we shall see in § 5, the corresponding changes to be made in our present computation will imply a velocity-dependent self-

energy and a damping force on the electron from the field induced by it, not to be found in MAXWELL's equations for empty space.

Consider now a gas of free electrons and its behaviour in the presence of an electromagnetic field, the motion of the electrons being governed by classical laws and restricted only by the Pauli exclusion principle in the initial state; from LIOUVILLE's theorem it follows, then, that the exclusion principle is obeyed at any later time. This approximation is conveniently termed semi-classical, and it is based on similar ideas as the Thomas-Fermi treatment of static electric fields. We shall find that the semi-classical treatment in some cases gives other results than the familiar classical treatment, where the electrons are assumed to be at rest.

The distribution function of the electrons over momentum space and ordinary space is $f = f(\vec{p}, \vec{r}, t)$, where $\vec{p}$ is the kinetic momentum conjugate to $\vec{r}$. The initial time-independent distribution function in the absence of external fields is called $f_0(p, \vec{r})$. The behaviour in time of f is given by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \dot{\vec{p}} \cdot \operatorname{grad}_p f + \vec{v} \cdot \operatorname{grad}_r f = -\frac{f - f_0}{\tau},$$

where one assumes a trend towards the equilibrium state, determined by the right-hand side of the equation. This damping of the motion may be pictured as due to the resistance arising from collisions with the positive background in the gas. As a rule, we regard $1/\tau$ as infinitely small, in which case the damping term serves to give the retarded solutions of the equations of motion, i. e. of the field equations.

The difference f_1 between f and f_0 is supposed to be small, and further f_0 is assumed to be independent of r. In the equation governing the behaviour of f we neglect second order terms and obtain the simple equation of motion for $f_1 = f_1(\vec{p}, \vec{r}, t)$

$$\frac{\partial}{\partial t} f_1 + e\left(\vec{E} + \frac{1}{c}(\vec{v} \times \vec{B})\right) \cdot \operatorname{grad}_p f_0 + \vec{v} \cdot \operatorname{grad}_r f_1 = -\frac{f_1}{\tau}, \quad (2.1)$$

where $\vec{E}$ and $\vec{B}$ are the total microscopic field strengths, derivable from the potentials. The term containing B in the Lorentz force may be omitted, being perpendicular to $\operatorname{grad}_p f_0$.

We wish to write equation (2.1) in terms of its Fourier components in space and time. Therefore we make the development $f_1(\vec{p}, \vec{r}, t) = \sum_{k,\omega} f_1(\vec{p}, \vec{k}, \omega) \exp(i\vec{k} \cdot \vec{r} - i\omega t)$, and similarly for the fields. We solve with respect to the Fourier components of f_1 and find

$$f_1(\vec{p}, \vec{k}, \omega) = \frac{1}{-i\vec{v} \cdot \vec{k} + i\omega - 1/\tau} e\vec{E} \cdot \operatorname{grad}_p f_0(p). \qquad (2.2)$$

The equation may be applied to, e. g., a Maxwell distribution, but we shall be concerned mainly with the case of complete degeneracy, where $\operatorname{grad}_p f_0 = -(\vec{p}/p) \cdot \delta(p - p_0)$, p_0 being the momentum at the surface of the Fermi distribution. Since f_0 was assumed independent of r, the same holds for p_0. We note that, for temperatures different from zero, the δ-function in $\operatorname{grad}_p f_0$ is replaced by a function of finite width, but as long as the temperature remains low compared to the degeneracy temperature there will be no appreciable change in the calculations below.

The current induced in the system can be expressed now as a function of the field. Multiplying f_1 in (2.2) by ev, and integrating over velocity space, we have for the transverse part of the induced electric current density

$$\vec{j}^{tr}_{\text{ind}}(\vec{k}, \omega) = \frac{2e^2}{h^3 c} \int d^3p \frac{\omega \vec{v}}{\vec{v} \cdot \vec{k} - \omega - i/\tau} \cdot \frac{\delta(p - p_0)}{p} \left(\vec{p} \cdot \vec{A}^{tr}(\vec{k}, \omega)\right).$$

The integration over p amounts to an integration over angles. When this is performed we compare with (1.9) and obtain the expression for the transverse dielectric constant of a degenerate free gas

$$\left.\begin{aligned} \varepsilon^{tr}(k, \omega) &= 1 + \frac{3\omega_0^2}{4\omega} \cdot \frac{m v_0}{p_0} \\ \cdot \left\{ -2 \frac{\omega + i/\tau}{v_0^2 k^2} + \frac{1 - (\omega + i/\tau)^2 / v_0^2 k^2}{v_0 k} \log \frac{v_0 k - \omega - i/\tau}{-v_0 k - \omega - i/\tau} \right\}, \end{aligned}\right\} \qquad (2.3)$$

where ω_0 is the classical resonance frequency of the electron gas,

or $\omega_0^2 = 4\pi e^2 \varrho/m$, ϱ being the density of electrons; v_0 is the velocity at the top of the Fermi distribution.

In quite a similar manner the induced charge density may be calculated from (2.2) as a function of the electric potential. The longitudinal dielectric constant in (1.9) is then found to be

$$\varepsilon^l(k, \omega) = 1 + \frac{3\omega_0^2}{v_0^2 k^2} \cdot \frac{m v_0}{p_0} \left\{ 1 + \frac{\omega + i/\tau}{2 k v_0} \log \frac{v_0 k - \omega - i/\tau}{-v_0 k - \omega - i/\tau} \right\}. \quad (2.4)$$

In these equations, and in the following, the logarithms denote principal values, so that their imaginary parts are between $-i\pi$ and $+i\pi$. It is seen that a change of sign of ω in (2.3) and (2.4) will have the same result as a change of sign in the damping term i/τ. Further, the dielectric constants depend on k only through the square of this vector.

We have distinguished between the momentum p_0 and mv_0. This discrimination is not necessary in the simple calculations in this paragraph, and in the following it is not stressed. However, a discrimination is useful for a relativistic gas, where (2.4) and (2.5) can be applied directly. Moreover, it is important when the self-energy influences the connection between p and v, as discussed in § 5.

As mentioned, a main difference between the treatment by Bohm and co-workers, and the present discussion, arises from a difference in attitude in the description of the field, to which we ascribe a precise classical meaning. While we obtain a common description of all waves of the field, Bohm treats essentially only long waves, for which a simple development in powers of $v_0^2 k^2/\omega^2$ can be made. Bohm asserts that the field equations represent merely collective motions, and not individual particle behaviour.

A simplification of the field equations corresponding to (2.3) and (2.4) is desirable and possible in most problems. We see that the dielectric constants depend on the dimensionless ratio between $|\omega + i/\tau|$ and $v_0 k$. If we are concerned with a field of given frequency the significant wave vectors k will collect around a certain corresponding vector, and the above dimensionless ratio may be said to have an approximate value, determined implicitly by the field equations. We can here distinguish between

two essentially different cases, namely $|\omega + i/\tau|$ larger or smaller than $v_0 k$. If it is assumed that the collision time τ is long, these two cases imply that during one period of the field the path travelled by an electron is, respectively, shorter and longer than the wavelength of the field. In the former case, the electron gets the impression of a time-dependent field while, in the latter case, it reacts nearly as it would in a static field.

Consider first the simpler case of k tending to zero. The transverse dielectric constant (2.3) then approaches the classical one for a homogeneous medium (omitting the factor mv_0/p_0),

$$\varepsilon^{tr} \cong 1 - \frac{\omega_0^2}{\omega\,(\omega + i/\tau)}\,, \quad \text{for} \quad v_0 k < |\omega + i/\tau|, \tag{2.5}$$

the first correction term being $(-\omega_0^2 k^2 v_0^2)/(5\,\omega\,(\omega + i/\tau)^3)$. In this limit, the expression for the longitudinal dielectric constant, (2.4), becomes of similar type,

$$\varepsilon^{l} \simeq 1 - \frac{\omega_0^2}{(\omega + i/\tau)^2}\left\{1 + \frac{3\,k^2 v_0^2}{5\,(\omega + i/\tau)^2} + \cdots\right\}, \quad \text{for} \quad v_0 k < |\omega + i/\tau|, \tag{2.6}$$

where we have included the first correction term in k^2. If one is not concerned with a degenerate gas, one can find this term simply by replacing $3v_0^2/5$ by the average of the velocity squared in the gas in question. It should be mentioned that the dispersion formula (2.6) has been discussed previously by several authors (cf., e. g., Bohm and Pines, 1952).

In the opposite limit of $v_0 k$ large compared with $\omega + i/\tau$ one finds a striking departure from the classical result. As mentioned, the electrons behave nearly as if the field were stationary, but in the dielectric constants finite imaginary terms appear, corresponding to energy absorption by the medium. Using (2.3) and (2.4), we obtain in this case (omitting again the factor mv_0/p_0)

$$\varepsilon^{tr} \cong 1 - \frac{\omega_0^2}{\omega^2}\left\{\frac{3\,\omega^2}{k^2 v_0^2} - i\,\frac{3\,\pi\omega}{4\,k v_0}\left(1 - \frac{\omega^2}{k^2 v_0^2}\right)\right\}, \quad \text{for} \quad v_0 k > |\omega + i/\tau|, \tag{2.7}$$

$$\varepsilon^{l} \cong 1 + \frac{3\,\omega_0^2}{k^2 v_0^2}\left\{1 - \frac{\omega^2}{k^2 v_0^2} + i\,\frac{\pi\omega}{2\,k v_0}\right\}, \quad \text{for} \quad v_0 k > |\omega + i/\tau|. \tag{2.8}$$

In approximate calculations one can use the formulae (2.5), (2.7) and (2.6), (2.8), instead of the more involved ones, (2.3) and (2.4). It is of interest, further, that in a qualitative sense one may apply, e. g., formula (2.5) for all values of k and ω, if only in this equation i/τ is replaced by $i/\tau + i\,4\,kv_0/3\,\pi$.

We may mention briefly some results concerning the non-degenerate gas. For a Boltzmann distribution, an approximate description can be obtained from the above formulae when replacing v_0 by the temperature velocity of the particles. More accurate expressions are found by introducing the distribution function $f_0 = C \cdot \exp(-mv^2/2\,\theta)$ in (2.2). It is apparent that, for ω large compared to $k\,(2\,\theta/m)^{1/2}$, the equations (2.5) and (2.6) are not far wrong; in the series development one need only introduce the proper averages of the particle velocities. In the opposite limit we get, e. g., for the longitudinal dielectric constant, corresponding to (2.8),

$$\varepsilon^l \simeq 1 + \frac{m\,\omega_0^2}{k^2\,\theta}\left\{1 + i\,\frac{\omega}{k}\left(\frac{\pi\,m}{2\,\theta}\right)^{\frac{1}{2}}\right\}, \quad \text{for } (2\,\theta k^2/m)^{\frac{1}{2}} > \left|\omega + i/\tau\right|, \quad (2.9)$$

where the first term in the brackets gives the Debye-Hückel formula for static electric fields, while the second term accounts for the energy absorption.

It is noteworthy that for a Boltzmann gas the imaginary part of the dielectric constants is finite, independently of the values of k and ω. Thus, for transverse fields, one finds directly from (2.2)

$$\mathrm{Im}\,(\varepsilon^{tr}) = \frac{4\,\pi}{\omega}\,\sigma^{tr} = \frac{\omega_0^2}{\omega\,k} \cdot \left(\frac{\pi\,m}{2\,\theta}\right)^{\frac{1}{2}} \cdot \exp(-m\,\omega^2/2\,\theta k^2), \quad (2.10)$$

the formula holding for sufficiently large values of τ.

Spin contribution to magnetic properties.

So far, we have considered charged particles without intrinsic magnetic moments. But already in the semi-classical treatment it is possible to include the electron spin in a simple manner. Consider for this purpose a single Fourier component $\vec{B}\,(\vec{k}, \omega)$ of the magnetic field. An electron with magnetic moment β has the interaction energy

$$\pm\,\beta B\,(\vec{k}, \omega)\,\exp(i\vec{k}\cdot\vec{r} - i\,\omega\,t),$$

corresponding to its moment being either parallel or antiparallel to the field. If the distribution function is f^+ and f^- for parallel and antiparallel spin, respectively, we get to the first order the following Boltzmann equations

$$(-i\omega + \vec{v}\cdot i\vec{k} + 1/\tau)\, f_1^{\pm}(\vec{k},\omega) \pm \beta B(\vec{k},\omega)\, i\vec{k}\cdot \mathrm{grad}_p f_0 \, m = 0, \quad (2.11)$$

where, for instance, $f_1^+(k,\omega) = f^+(k,\omega) - f(k,\omega)$, f being the spin independent distribution function in (2.1). The contributions to the induced magnetic moment of the gas from the parallel and antiparallel distributions are equal. We find readily, for the Fourier components of the spin magnetic moment per unit volume,

$$\vec{M}(\vec{k},\omega) = \sum_p m\beta^2 \vec{B}(\vec{k},\omega) \frac{\vec{k}\cdot \mathrm{grad}_p f_0}{\vec{v}\cdot\vec{k} - i/\tau - \omega}.$$

A magnetic moment $\vec{M}$ is equivalent to a current $(c/4\pi)$ rot $\vec{M}$, and thus we find, by summation, the spin contribution to the transverse dielectric constant of a degenerate gas

$$\delta\varepsilon^{tr}(k,\omega) = -\frac{k^2 p_0^2 c^2 \beta^2}{\pi^2 \omega^2 \hbar^3 v_0}\left\{1 + \frac{\omega + i/\tau}{2kv_0}\log\frac{kv_0 - \omega - i/\tau}{-kv_0 - \omega - i/\tau}\right\}. \quad (2.12)$$

If, more naturally, we describe the effect as a contribution to the permeability μ we can use the relation $\delta(1/\mu) = -(\omega/kc)^2\,\delta\varepsilon^{tr}$. In the limit of low frequencies we obtain

$$\delta\left(\frac{1}{\mu}\right) \simeq -\frac{p_0^2\beta^2}{\pi^2\hbar^3 v_0}\left\{1 + i\frac{\pi\omega}{2kv_0}\right\}. \quad (2.13)$$

For a free electron gas, the first term in the brackets leads to the familiar spin paramagnetism, equal to three times the diamagnetic contribution. Since, generally, the spin contribution to the electromagnetic properties of the system is small compared to the orbital contributions (2.3), it will be neglected in the following.

When comparing our description in this paragraph with treatments by other authors it should be mentioned, first of all, that a hydrodynamical discussion of the motion of a Thomas-Fermi gas has been attempted by Bloch (1933, 1934). The pressure—proportional to $\varrho^{5/3}$—arising from the zero-point kinetic energies was introduced in hydrodynamical equations of motion. Using only the two parameters pressure and velocity of the liquid model for the description of the state of the electron gas, Bloch

obtained an approximate and smoothed-out picture of the motion. It is easily shown that, in the linear approximation, the model of BLOCH gives, for the dielectric constant of the longitudinal motion,

$$\varepsilon^l = 1 - \frac{\omega_0^2}{(\omega + i/\tau)^2 - k^2 u^2}, \tag{2.14}$$

where the constant u^2 is given by $u^2 = (5/9) \cdot v_0^2$ and thus comparable to the velocity, v_0, at the top of the Fermi distribution. The velocity u may be said to represent the sound velocity in the gas without interaction ($\omega_0 = 0$). When kv_0 is small compared to ω we make a series development of (2.14) and compare with (2.6). In this limit, the latter formula leads to $u^2 = (3/5) \cdot v_0^2$, which is close to the above value found by BLOCH (cf. BOHM and GROSS, 1949). For values of kv_0 larger than ω, equation (2.14) gives a screening of the field quite similar to that in (2.8), although the value of u^2 should now be changed to $(1/3) \cdot v_0^2$; moreover, the important finite imaginary part of the dielectric constant (2.8) is not reproduced in (2.14). We shall return, presently, to some of the simple features described by (2.14).

As regards the application of the hydrodynamical model of BLOCH to the transverse field, we note that the transverse motions do not affect the density. The model must therefore give the same result as the simplest classical picture. Accordingly, one finds the transverse dielectric constant (2.5), which formula is obtained from (2.3) when putting the electronic velocity v_0 equal to zero. The hydrodynamical model is not appropriate, apparently, when (2.7) applies, i. e. for $kv_0 > |\omega + i/\tau|$. It fails to describe the orbital diamagnetism and the anomalous skin effect (cf., e. g., LINDHARD, 1953).

As mentioned in the introduction, an interesting attempt to give a more exact solution of the equations of motion for a degenerate gas has been made recently by TOMONAGA (1950). His treatment, however, was limited to the one-dimensional case. On the assumption that the field contained only waves of long wavelength, which in the present formulation means that semi-classical methods apply, TOMONAGA obtained a solution without recourse to usual perturbation theory. The result was a linear field equation for the longitudinal field given by the

dielectric constant (2.14) of BLOCH's three-dimensional model, only now with $u = v_0$, v_0 being the velocity at the top of the one-dimensional Fermi distribution. Now, it so happens that, if we apply the present method to the one-dimensional case where the velocity v and the wave vector can be only parallel and anti-parallel, we find immediately from (2.2) the dielectric constant of TOMONAGA, (2.14) with $u = v_0$. It may appear surprising that a first order treatment gives the same result as a more precise transformation. The reason for the agreement is that in the one-dimensional case the first order calculation is in fact self-consistent if only long waves are considered, as will be shown in § 5. We note, further, that the Tomonaga treatment is not an exact one, because of the omission of the short waves of the field. In § 3 we shall deduce an expression, (3.12), for the one-dimensional dielectric constant, based on quantum mechanical equations of motion for the electrons.

Let us regard, for a moment, the simple picture suggested by the Bloch model (2.14). For long waves the energy of the field quanta is

$$E = (p^2 u^2 + \hbar^2 \omega_0^2)^{\frac{1}{2}} \simeq \mu u^2 + p^2/2\mu ,$$

where $p = \hbar k$ is the momentum, and μ may be described as the mass of the quanta. The value of μ is

$$\mu = \frac{\hbar \omega_0}{u^2} = 2m \left(\frac{e^2}{3\pi \hbar v_0} \right)^{\frac{1}{2}} \cdot \left(\frac{v_0}{u} \right)^2 ,$$

which is usually of the order of the electron mass. If, for instance, a charged particle passes through the gas, such field quanta, equivalent to particles, will be created. In the first approximation, these particles will subsist in the system, but a closer inspection shows that they have a finite life-time (cf. § 5). For increasing values of p we find, according to the more precise field equation (2.4), that the solution of the field equations is considerably more complicated, and large damping terms appear. In this region field quanta similar to particles can not be identified. However, for nearly stationary cases the field equations, (2.8), are of the Yukawa type with screening length $1/\varkappa = v_0/3\,\omega_0$ (equation (2.14) gives $1/\varkappa = u/\omega_0$). In BLOCH's model the screening length

is equal to the Compton wavelength, $\hbar/\mu u$, of the field quanta, and in the more accurate description this is approximately true. The properties of the field quanta, therefore, are not unlike those of mesons. Yet, it should be emphasized that the field quanta in (2.14) account only for part of the properties of the gas, excepting the one-dimensional case. If we consider the thermal properties of the system (2.14), we find that the excitation is vanishing, unless the temperature, θ, is of the order or larger than $\hbar\omega_0$. This differs from the expected result for a free gas, i. e. thermal energy proportional to θ^2. When one uses instead (2.8), together with the particle equations, a θ^2-dependence is found, as will be discussed in II.

In a treatment of stopping problems KRONIG and KORRINGA (1943), and KRONIG (1949), have attempted to describe the motion of a metallic electron gas as that of a charged liquid, subject to friction from the background of positive ions and having a certain internal viscosity. The pressure effects, important in the model of BLOCH, are neglected. For comparison, we shall quote the longitudinal dielectric constant corresponding to the model of KRONIG and KORRINGA.

Let ϱ_m and ϱ_c be the densities of mass and charge of the liquid. The viscosity is η, and the friction with the static background is ξ. From the classical hydrodynamical equations of motion one finds then, in the linear approximation,

$$\varepsilon^l = 1 - \frac{4\pi\varrho_c^2}{\varrho_m\omega^2 + i\omega(\xi + 2\eta k^2)}. \tag{2.15}$$

This differs from the classical ε in (2.5) only by the term involving the viscosity η. If this arbitrary parameter is large, as assumed by KRONIG and KORRINGA, one gets a strong dependence of ε^l on the wave number of the field. However, for large values of k, where the viscosity term is dominating, quantum theory takes over; in § 3 will be given a simple and more appropriate formula for ε^l at short wave lengths (see, for instance, eq. (3.4) and page 32). The model contained in (2.15) is less appropriate than that of BLOCH, and the picture involving a viscosity of the kind met with in ordinary hydrodynamics can be somewhat misleading (cf. p. 39). We note that the transverse dielectric constant in the model of KRONIG and KORRINGA will be quite similar to (2.15).

Already here mention may be made of the field equations proposed by LONDON for superconductors. It can be of interest to compare his equations with those in the present treatment of a free gas. We are then concerned with the equations (I)—(VIII), p. 29. F. LONDON (1950). They are seen to result from the following simple dielectric constants

$$\varepsilon^{tr} = \varepsilon^{l} = 1 - \frac{4\pi}{\Lambda\omega^2} + i\,\frac{4\pi\sigma_n}{\omega}, \tag{2.16}$$

where Λ is a constant which, according to LONDON, defines a density of superconducting electrons, $\varrho_s = (m/e^2\,\Lambda)$. Further, $\sigma_n = \varrho_n e^2 \tau_n/m$ is the conductivity of the remaining normal electrons. It is not easy to understand why ε^{tr} and ε^{l} should be put equal, corresponding to $\mu = 1$. For this reason, and because (2.16) is not specified in detail for long and short waves of the field, it is difficult to make a direct comparison with the dielectric constants derived above.

It can be illustrative to compare (2.16) with the conventional picture of an ideal conductor. The field equations for the latter are usually assumed to be those resulting from (2.5) with $1/\tau = 0$. Therefore, for fields varying not too quickly in time, LONDON's equations are merely the retarded equations for the usual model of an ideal conductor. It appears from (2.7) and (2.8) that the mentioned model of an ideal conductor is inadequate as a picture of an electron gas, which makes more obscure the meaning of equations as (2.16) (cf. II and LINDHARD, 1953).

§ 3. Quantum mechanical treatment of electrons.

We shall now include the quantum mechanical description of the electronic motion in the calculation which, otherwise, follows quite similar lines as the semi-classical treatment in § 2. The main difference is that previously we distinguished only between wavelengths $1/k$ of the field, long or short compared to v_0/ω, while now we must further compare $1/k$ with the wavelengths $1/k_0 = \hbar/mv_0$ of the electrons in the gas. As long as one is concerned with long waves of the field there will be only minor corrections to the semi-classical formulae, but for field waves shorter than $1/k_0$ the field equations are completely changed.

In the calculations we make the same assumption as in § 2 regarding a common field in which the electrons move. This means essentially that time-dependent Hartree equations are used. As to the initial state we take, as before, simply the freely moving electrons of the Sommerfeld model and compute to first order in e^2. None the less, our results go beyond first order when we introduce an effective electron mass (cf. § 5).

Already here mention may be made of one particular circumstance concerning the behaviour in time of a system when disturbed by external fields. It is apparent that the wave functions will not develop independently in time, because each particle moves in a field determined by the others. But, since the originally

orthogonal one-particle wave functions are governed all by the same one-particle Hamiltonian, they will automatically remain orthogonal. This has the particular advantage that the exclusion principle need not be taken into account explicitly in the dynamical treatment, just as in the semi-classical case. It follows, moreover, that for a wide class of phenomena, e. g. in impacts by external charges, the seeming reduction in transition possibilities due to the exclusion principle* does not come into play.

As regards the boundary conditions in the perturbation treatment, one should at the time $t = -\infty$ find the unperturbed state. This condition is fulfilled if an infinitesimal negative imaginary term, $-i\Gamma/2$, is introduced in the energies of the electron states, because one then can obtain the retarded solutions of the equations of motion. It can be convenient to assume that the width Γ of the excited states is finite, since the width is connected with the resistance in simple pictures of metals. However, this manner of describing resistance is sometimes too crude,—it is less justified than in the classical equations (2.1). The way in which the metallic resistance appears in the field equations belongs to the properties of the positive background of charge, with which we are not immediately concerned. In such more complicated cases the use of causal dielectric constants is to be preferred (cf. § 5).

We consider an assembly of electrons having one-particle wave functions $\psi_n^0(\vec{r}, t) = \varphi_n^0(\vec{r}) \cdot \exp(-i\omega_n t + \Gamma_n t/2\hbar)$, the system being initially e. g. in the ground state. The perturbing term in the Hamiltonian is

$$\Omega = e\Phi(\vec{r}, t) + \frac{e}{2mc}\left(\vec{p} \cdot \vec{A}(\vec{r}, t) + \vec{A}(\vec{r}, t) \cdot \vec{p}\right), \qquad (3.1)$$

where for the present we disregard spin contributions.

Equations for longitudinal field.

The perturbation gives rise to an induced charge and current density for each of the electrons. The total induced charge density, $\varrho_i(\vec{r}, t)$, in the system is the sum of the induced densities for the individual electrons. We write, as before, the fields and the

* Explicit reductions of such kind have been applied in nucleon problems by, e. g., Goldberger (1948), and Blatt and Weisskopf (1952).

charge densities in Fourier components in space and time. Choosing the gauge to be such that the longitudinal vector potential vanishes, we find from (1.9) the connection between the longitudinal dielectric constant, $\varepsilon^l(k, \omega)$, and the induced charge density. We specialize to the case of free electrons with wave functions $\varphi_n^0(\vec{r}) = V^{-1/2} \exp(i\vec{k}_n \cdot \vec{r})$ for the initial states, and similarly for the final states. When we choose a Fourier component $\vec{k}, \omega$ of the induced charge density, an electron with momentum $\hbar\vec{k}_n$ can only jump to a state with momentum $\hbar(\vec{k}_n \pm \vec{k})$. The contribution from this electron to the Fourier component of the induced charge density is, assuming a constant imaginary energy difference, $i\gamma$,

$$\varrho_{\text{ind.}}(\vec{r}, t) = -\frac{2me^2}{\hbar^2 V}\left\{\frac{\Phi(\vec{k}, \omega)\cdot \exp(i\vec{k}\cdot\vec{r} - i\omega t)}{k^2 + 2\vec{k}\cdot\vec{k}_n + \frac{2m}{\hbar}\left(-\omega - i\frac{\gamma}{\hbar}\right)} + \text{compl. conj.}\right\}. \quad (3.2)$$

The distribution function of the electrons we denote as $f = f(E_n)$, E_n being the unperturbed energies. Summing (3.2) over the electrons and using the definition of the retarded dielectric constant, we find for free electrons

$$\varepsilon^l(k, \omega) = 1 + \frac{2m^2\omega_0^2}{\hbar^2 k^2}\sum_n \frac{f(E_n)}{N}\left\{\frac{1}{k^2 + 2\vec{k}\cdot\vec{k}_n - \frac{2m}{\hbar}\left(\omega + i\frac{\gamma}{\hbar}\right)}\right. \left. + \frac{1}{k^2 - 2\vec{k}\cdot\vec{k}_n + \frac{2m}{\hbar}\left(\omega + i\frac{\gamma}{\hbar}\right)}\right\}, \quad (3.3)$$

where $N = \sum_n f(E_n)$ is the total number of electrons, and ω_0 as before is the classical resonance frequency of the gas, $\omega_0^2 = 4\pi e^2 N/mV$.

In (3.3), we sum freely over all electron states, since contributions from jumps between two occupied states cancel, as follows from our previous remarks. In the derivation of (3.3), the technique of quantum field theory is not advantageous, but can be so in higher order computations.

We apply first the formula (3.3) to the extremely simple case of the electrons being initially at rest. Accordingly, the momenta $\hbar k_n$ are neglected in (3.3), which leads to

$$\varepsilon^l(k,\omega) = 1 + \frac{\omega_0^2}{\frac{\hbar^2}{4\,m^2}\,k^4 - \left(\omega + i\frac{\gamma}{\hbar}\right)^2} \quad . \tag{3.4}$$

This formula corresponds to the classical result (2.6) and shows in a direct manner how the inclusion of quantum effects implies a strong dependence on the wavelength of the field.

Take next the case of the Fermi distribution, and assume $f(E_n) = 1$ for $E_n \leq E_0$, while for higher energies $f(E_n) = 0$. Put further $E_0 = mv_0^2/2 = \hbar^2 k_0^2/2\,m$. In the summation in (3.3) we first average over the electrons with the same energy $\hbar^2 k_n^2/2\,m$, that is to say, we integrate over the angle between k_n and the fixed vector k. One then obtains characteristic logarithmic expressions, not unlike those in the semi-classical formulae for ε. Finally, we integrate over the energy E_n, from 0 to E_0. When quoting the resulting formula it is convenient to use a few abbreviations; we write $z = k/2\,k_0$, $\omega' = \omega + i(\gamma/\hbar)$, $u = |\,\omega\,|/kv_0$, and $u' = \omega'/kv_0$. The parameter z, which contains the wavelength of the field and that of an electron in the gas, serves to indicate how far one is away from the classical limit, the latter corresponding to $z = 0$. Further, the quantity u (or u') shows whether an electron during one period of the field travels a distance longer or shorter than one wavelength of the field, these two cases being represented by respectively $u < 1$ and $u > 1$. The formula for the dielectric constant of the degenerate Fermi gas is then

$$\varepsilon^l = 1 + \frac{3\,\omega_0^2}{k^2 v_0^2} \cdot f, \tag{3.5}$$

$$\left.\begin{aligned} f = \frac{1}{2} + \frac{1}{8z}\{1 - (z-u')^2\}\log\frac{z-u'+1}{z-u'-1} \\ + \frac{1}{8z}\{1 - (z+u')^2\}\log\frac{z+u'+1}{z+u'-1}, \end{aligned}\right\} \tag{3.6}$$

where principal values are taken. In the present connection, we

are concerned with very small values of the damping γ. In the limit of small γ we find the following expressions for the real and imaginary parts of $f = f_1 + if_2$,

$$\left.\begin{aligned} f_1(u,z) = \frac{1}{2} + \frac{1}{8z}\{1-(z-u)^2\}\log\left|\frac{z-u+1}{z-u-1}\right| \\ + \frac{1}{8z}\{1-(z+u)^2\}\log\left|\frac{z+u+1}{z+u-1}\right|, \end{aligned}\right\} \quad (3.7)$$

$$f_2(u,z) = \left\{\begin{array}{ll} \dfrac{\pi}{2}u, & \text{for} \quad z+u<1, \\ \dfrac{\pi}{8z}\{1-(z-u)^2\}, & \text{for} \quad |z-u|<1<z+u, \\ 0, & \text{for} \quad |z-u|>1. \end{array}\right\} \quad (3.8)$$

The contents of these formulae for the longitudinal dielectric constant are perhaps best appreciated by finding the approximate values in a few limiting cases. It is seen that, for $z = 0$, the equations (3.5), (3.6) lead to the semi-classical formula (2.4). A similar result is obtained for small values of z. Consider here the case of $z + u < 1$. This corresponds in the semi-classical treatment to formula (2.8), where $u < 1$ and thus the electrons can move through several wavelengths of the field during one period. We assume that the damping is small and develop (3.7) in powers of the small quantities $u \pm z$. This leads to

$$\varepsilon^l(k,\omega) = 1 + \frac{3\omega_0^2}{v_0^2 k^2}\left\{1 + i\frac{\pi\omega}{2v_0 k} - \frac{\omega^2}{v_0^2 k^2} - \frac{k^2}{12k_0^2} + \cdots\right\}, \quad z+u<1 \quad (3.9)$$

a formula equivalent to (2.8), except for the last term in the brackets, giving rise to the subtraction of a constant in ε^l (cf. also LINDHARD, 1946).

Another case with correspondence to the treatment in § 2 is that of $u > 1 + z$. This inequality implies that $\hbar\omega$ is large compared with the energy transfer $\hbar^2(k^2 + 2kk_0)/2m$ to an electron. Therefore the classical formula on the form (2.6), where $u > 1$, must apply, as is also seen from a series development of (3.6)

$$f(u', z) = -\frac{1}{3\,u'^2} - \frac{1}{5\,u'^4} - \frac{z^2}{3\,u'^4} \cdots, \quad |u'| > 1 + z. \tag{3.10}$$

In the more typical quantum mechanical case where $z \gtrsim u + 1$, the longitudinal dielectric constant tends to 1 for increasing z, because the electrons respond only weakly to the short waves of the field. We shall quote only the result for the region where the imaginary part of ε^l is finite. Thus, we find

$$\varepsilon^l(k, \omega) = 1 + \frac{3\,\omega_0^2}{v_0^2\,k^2}\left\{\frac{i\pi}{8\,z}(1 - (z-u)^2) + \frac{z-u}{2\,z} + \cdots\right\}, \tag{3.11}$$

in the case of $|u - z| < 1$, $u + z > 1$. The resulting resonance effect is of the same kind as that in (3.4), for large values of k.

Before finishing this discussion of the longitudinal field we shall deduce the dielectric constant for the case of a Fermi gas in one dimension. In § 2, it was shown that the semi-classical treatment gives just the field equations of TOMONAGA (1950). The present quantum mechanical perturbation method must of course give a somewhat different result for short waves of the field.

The desired equations can be derived immediately from equation (3.3), if there we put equal to zero all vector components in the direction of the y- and z-axes. As before, it is supposed that $f(E_n) = 1$ for $E_n < E_0$, and otherwise $f = 0$. Here, $E_0 = mv_0^2/2 = \hbar^2 k_0^2/2\,m$. Summing over $k_n = k_{xn}$ one finds that

$$\varepsilon^l(k, \omega) = 1 + \frac{m^2\,\omega_0^2}{2\,\hbar^2\,k^3\,k_0}\left\{\log\frac{z - u' + 1}{z - u' - 1} + \log\frac{z + u' + 1}{z + u' - 1}\right\}, \tag{3.12}$$

which formula for large k_0 tends to the semi-classical one, (2.14), with $u = v_0$. As usual, principal values are to be taken in the logarithmic expression in (3.12). In the other extreme of k_0 small compared to the wave vector k of the field, the formula (3.12) leads to the general result (3.4). It is seen that, while in the semi-classical treatment the case of one dimension was much simpler than the three-dimensional case, this is not so in the quantum mechanical treatment.

Equations for transverse field.

For the case of transverse fields we shall apply the same perturbation method as for longitudinal fields, and again it is assumed that the motion of the electrons is non-relativistic. The perturbation term in the Hamiltonian is (3.1), where the spin contribution is neglected. The inclusion of spin would only give rise to small changes in the field equations, as is evident from the result (2.13). In analogy to (3.2) we now find the contribution to the induced transverse current from an electron with wave vector $\vec{k}_n$, in the simple harmonic field $\vec{A}(\vec{k}, \omega) \cdot \exp(i\vec{k} \cdot \vec{r} - i\omega t)$ + compl. conj., corresponding to a single Fourier component,

$$\left.\begin{aligned} &\delta\vec{j}_{ind}(\vec{r}, t) = \\ &-\frac{e^2}{2mcV}\left\{\left(2\vec{k}_n + \vec{k}\right)\frac{\left(2\vec{k}_n + \vec{k}\right)\cdot\vec{A}(k,\omega)}{k^2 + 2\vec{k}\cdot\vec{k}_n - \frac{2m}{\hbar}\left(\omega + i\frac{\gamma}{\hbar}\right)} + 2\vec{A}(k,\omega)\right\} \\ &\cdot \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right) + \text{compl. conj.} \end{aligned}\right\} \qquad (3.13)$$

We introduce the distribution function of the electrons and, applying (1.3), we can express the transverse dielectric constant on a closed form. The dielectric constant is a three-dimensional tensor which, because of the symmetry in problem, is on diagonal form, with diagonal elements

$$\left.\begin{aligned} &\varepsilon^{tr}(k, \omega) = 1 \\ &\frac{2\omega_0^2}{\omega^2}\sum_n \frac{f(E_n)}{N}\left\{\left(k_n^2 - \left(\vec{k}\cdot\vec{k}_n\right)^2/k^2\right)\left(\frac{1}{k^2 + 2\vec{k}\cdot\vec{k}_n - \frac{2m}{\hbar}\left(\omega + i\frac{\gamma}{\hbar}\right)}\right.\right. \\ &\left.\left. + \frac{1}{k^2 - 2\vec{k}\cdot\vec{k}_n + \frac{2m}{\hbar}\left(\omega + i\frac{\gamma}{\hbar}\right)}\right) - \frac{1}{2}\right\}. \end{aligned}\right\} \qquad (3.14)$$

We pass immediately to the summation for the case of a degenerate Fermi gas, which will give the quantum mechanical equation corresponding to (2.3). With fixed values of ω and the

vector $\vec{k}$ we first average over the angle between $\vec{k}_n$ and $\vec{k}$, and next integrate over the numerical value of $\vec{k}_n$, from 0 to k_0. Using a familiar notation we can then write

$$\varepsilon^{tr}(k, \omega) = 1 - \frac{\omega_0^2}{\omega^2} f^{tr}(u, z), \tag{3.15}$$

where $f^{tr} = f_1^{tr} + i f_2^{tr}$ is determined from the equations

$$f_1^{tr}(u, z) = \frac{3}{8}(z^2 + 3u^2 + 1) - \frac{3}{32 z}\left\{(1 - (z-u)^2)^2 \log\left|\frac{z-u+1}{z-u-1}\right| + (1 - (z+u)^2)^2 \log\left|\frac{z+u+1}{z+u-1}\right|\right\}, \tag{3.16}$$

$$f_2^{tr}(u, z) = \begin{cases} \frac{3}{4}\pi u (1 - u^2 - z^2), & \text{for } u + z < 1, \\ \frac{3\pi}{32 z}(1 - (u-z)^2)^2, & \text{for } |u - z| < 1 < u + z, \\ 0, & \text{for } |u - z| > 1. \end{cases} \tag{3.17}$$

The equations (3.16), (3.17) are valid in the limit of $\gamma \to 0$. The formula for f, when γ is finite, can easily be obtained from (3.16) by comparing this equation with (3.7) and (3.6).

Let us find the values of the above functions in the simple limiting cases introduced earlier. Suppose first that $u + z < 1$. It is seen that here

$$\varepsilon^{tr}(k, \omega) = 1 - \frac{\omega_0^2}{\omega^2}\left\{\frac{k^2}{4 k_0^2} + \frac{3\omega^2}{k^2 v_0^2} + \cdots - i\frac{3\pi\omega}{4 k v_0}\left(1 - \frac{\omega^2}{k^2 v_0^2} - \frac{k^2}{4 k_0^2}\right)\right\}, \quad u + z < 1, \tag{3.18}$$

This result is not unlike that in the semi-classical case, (2.7); only we find a characteristic new term—the first one in the brackets in (3.18). When multiplying by ω^2/k^2c^2, this term is seen to contribute $-(k_0/3\pi)\cdot(e^2/mc^2)$ to the permeability. It represents the weak Landau diamagnetism of the gas and is here equal to one third of the paramagnetic spin contribution (2.13). We note that the Landau term appears only in the field equations when $u + z \ll 1$, i.e. for small values of the frequency ω.

For the case $u > 1 + z$ we have again correspondence to the classical results, and the value of ε^{tr} is found from the series development

$$\varepsilon^{tr}(k, \omega) = 1 - \frac{\omega_0^2}{\omega^2}\left\{1 + \frac{1}{5\,u^2} + \frac{3\,z^2}{4\,u^2} + \cdots\right\},\ u > 1 + z. \quad (3.19)$$

Suppose that terms higher than the first in the brackets can be neglected. It is apparent, then, that for finite γ's there remains a small difference from the previous semi-classical equation (2.5), because the damping is not reproduced in (3.19). This difference has its origin in the term $(e^2/mc)\,\vec{A}\,(\vec{r}, t)\cdot|\psi|^2$ in the expression for the quantum mechanical current of a particle, and is due to the crude assumptions made regarding the damping in this paragraph. The difference disappears in a more systematic treatment, as outlined in § 5.

In the quantum mechanical limit the region of interest is $(u-z)^2 < 1 << (u+z)^2$, where the imaginary part of ε is finite,

$$\varepsilon^{tr}(k, \omega) = 1 - \frac{\omega_0^2}{\omega^2}\left\{1 - i\frac{3\,\pi}{32\,z}(1-(u-z)^2)^2\right\},\ |u-z| < 1 << u+z. \quad (3.20)$$

The present calculations of the dielectric constants were non-relativistic. As to a relativistic description we may quote one simple example. It is assumed that the electrons are originally at rest, and the induced charges and currents are computed from the Dirac equation for the electron. From a so-called simple calculation, one then finds that the longitudinal and transverse dielectric constants are equal and given by

$$\varepsilon^{l}(k, \omega) = \varepsilon^{tr}(k, \omega) = 1 + \frac{\omega_0^2}{\dfrac{\hbar^2 k^4}{4\,m^2} - \left(1 + \dfrac{\hbar^2 k^2}{2\,m^2 c^2}\right)\omega'^2 + \dfrac{\hbar^2\,\omega'^4}{4\,m^2\,c^4}}, \quad (3.21)$$

where $\omega' = \omega + i\dfrac{\gamma}{\hbar}$. It is evident that, for small k and ω, the formula leads to (3.4). This result, moreover, proves in a direct manner the equality of ε^{tr} and ε^{l} in the case represented by (3.4), where we calculated only the longitudinal dielectric con-

stant. An application of (3.21) can be found, e.g., in the question of energy loss by a relativistic particle.

§ 4. Damping of motion of charged particles, and self-energy.

The slowing-down of a charged particle moving through the system is a simple and interesting problem, in which the above description can be of great convenience. The motion of a charged particle is of particular interest in the present connection because it relates to the self-consistency of the method of approach employed in this paper, as will be discussed more closely in the next paragraph. We shall derive general formulae for, e.g., the specific energy loss. However, the dielectric constants calculated to first order in e^2, starting from the free particle approximation, will not always lead to very accurate results, and accordingly we do not perform elaborate computations of the stopping.

A treatment of stopping problems based on the dielectric properties of a medium was suggested early by FERMI (1924). As we shall see, a detailed discussion of the problem in this manner will not be confined to the simple dispersion properties of the medium. Still, if one considers only distant collisions, these properties are sufficient, which circumstance was utilized by FERMI (1940) in a treatment of polarization effects in the stopping of relativistic particles. We shall show that our classical field equations are applicable even for the short waves of the field.

It is not the purpose here to discuss the numerous aspects of the problem of energy loss in matter by fast charged particles; a general survey based on simple concepts has been given by N. BOHR (1948). For the present we are concerned with the case where a perturbation treatment can be used. Let it be assumed, further, that the particle is so heavy that its energy loss is comparatively small, and it behaves approximately as a classical point charge, ze, moving with a constant velocity, v, — we may for instance consider a proton. This simplification allows a description by forced classical vibrations, since the source density is given beforehand as a function of space and time.

Before specializing to a free electron gas we give a general

derivation of the energy loss suffered by the particle. In the case considered the charge density is

$$\varrho_0(\vec{r}, t) = z_1 e \delta(\vec{r} - \vec{v}t) = \frac{z_1 e}{V} \sum_{\vec{k}} \exp\left(i\vec{k} \cdot \vec{r} - i\vec{k} \cdot \vec{v}t\right). \quad (4.1)$$

The energy loss per unit path is $z_1 e$ times the electric field strength E at the immediate position $\vec{r} = \vec{v}t$ of the particle, which field is antiparallel to $\vec{v}$. Limiting at first the treatment to the longitudinal field, as is always sufficient if the particle velocity is non-relativistic, we obtain for the time-independent field to which the particle is subjected

$$\vec{E}(\vec{v}t, t) = -\sum_{\vec{k}} i\vec{k} \Phi\left(\vec{k}, \vec{k} \cdot \vec{v}\right), \quad (4.2)$$

which quantity therefore depends on the imaginary, or odd, part of $\Phi\left(\vec{k}, \omega\right)$. The resulting specific energy loss is obtained from (4.1), (4.2), and (1.4); we transform the three-dimensional integration over k to one over its numerical value and one over $\omega = kv \cos\left(\vec{k}, \vec{v}\right)$, and find*

$$\frac{dE}{dR} = -\frac{z_1^2 e^2}{\pi v^2} \operatorname{Im} \left\{ \int_0^{\infty} \frac{dk}{k} \int_{-kv}^{+kv} d\omega \frac{\omega}{\varepsilon^l(k, \omega)} \right\}, \quad (4.3)$$

where $\operatorname{Im}(x + iy) = y$. It is noteworthy that, since the dielectric constant enters in the denominator, the energy loss depends both on the longitudinal conductivity (1.10) and on the polarizability.

For the sake of completeness, we derive the contribution from the transverse field too. The transverse electric field is given by $\vec{E}^{tr}(\vec{k}, \vec{k} \cdot \vec{v}) = i(\vec{k} \cdot \vec{v}/c)\vec{A}^{tr}(\vec{k}, \vec{k} \cdot \vec{v})$. By means of (1.3) the field is obtained from the transverse part of the current corresponding to equation (4.1). It is then an easy matter to deduce the stopping

$$\left.\frac{dE}{dR}\right|_{tr} = -\frac{z_1^2 e^2}{\pi c^2} \cdot \operatorname{Im} \left\{ \int_0^{\infty} k dk \int_{-kv}^{+kv} \omega \, d\omega \frac{1 - \omega^2/k^2 v^2}{k^2 - \frac{\omega^2}{c^2} \varepsilon^{tr}(k, \omega)} \right\}. \quad (4.4)$$

Since the contribution from the longitudinal field is given always by (4.3), the total average energy loss is the sum of (4.3) and

* The energy loss may also be calculated from the energy dissipation term in the conservation equation for the energy of the field (the bracketed term on the right-hand side of (1.11)). This leads again to (4.3).

(4.4). In some cases (4.4) can be simplified considerably. Thus, if ε^{tr} has no finite imaginary part and can be considered as independent of k, the formula reduces to the one deduced by FRANK and TAMM (1937) for the Čerenkov radiation (see further A. BOHR, 1948). As mentioned above, we shall be concerned only with the case where the contribution (4.4) can be neglected.

It is perhaps not out of place to interpose here some further general remarks, as an introductory to the calculations below. In the problem of the stopping power of matter it can be illustrative to consider two extreme states of motion of the electrons in the substance. First, one can regard separated electrons, each bound in, say, a harmonic oscillator of frequency ω. This problem was solved at an early date (N. BOHR, 1913), and it is a characteristic feature that the energy transfers to the electrons become very small at distances exceeding the adiabatic limit v/ω, determined by the velocity of the particle and the frequency of the oscillator. Second, one may consider the opposite case where the electrons move freely in the system. Then, a different kind of reduction must appear in the energy transfer at large distances (if there were no reduction the total energy transfer would be infinite). The proper explanation is to be found in the screening of the field by polarization in the gas, as was shown by KRAMERS (1947), but not in the resistance damping, the latter having been suggested by v. WEIZSÄCKER (1933). Simply for dimensional reasons it is clear that the frequency which for free electrons replaces that of the harmonic oscillator must be the classical resonance frequency of the gas, $\omega_0 = (4\pi e^2 \varrho/m)^{\frac{1}{2}}$, where ϱ is the density of electrons. By combining the above two pictures—the harmonic oscillator and the free gas—in a suitable way, one should be able to account for the stopping effects in atomic systems. We note here that the free electron model—i.e. the Thomas-Fermi treatment—is useful even in the description of atoms. As a matter of fact, the present calculations on a free electron gas were utilized in a simple general discussion of atomic stopping power in a recent paper (LINDHARD and SCHARFF, 1953).

Let us apply the formula (4.3) to the simplest case: a homogeneous gas of electrons at rest. We therefore introduce (3.4) in (4.3) and find by integration

$$\frac{dE}{dR} = \frac{4\pi z_1^2 e^4}{mv^2} \cdot \varrho \cdot L, \tag{4.5}$$

where

$$L = \log \left\{ \frac{mv^2}{\hbar\omega_0} + \left(\left(\frac{mv^2}{\hbar\omega_0} \right)^2 - 1 \right)^{\frac{1}{2}} \right\}. \tag{4.6}$$

In the model used the energy loss therefore vanishes for $mv^2 \leqq \hbar\omega_0$. However, the model is preferably to be applied in the familiar extreme of large v. From a series development of (4.6) one here gets

$$L = \log \frac{2\, mv^2}{\hbar\omega_0}, \tag{4.7}$$

which is just the result of KRAMERS*. One may indeed, as indicated above, derive this formula using a simple qualitative argument (cf. A. BOHR, 1948; LINDHARD and SCHARFF, 1953), but then an undetermined constant remains inside the logarithm.

If one calculates separately the contribution to the energy loss from distant collisions, one can from (2.6) or the Bloch model (2.14) refind the result (4.7), again apart from a constant in the logarithm. Mention should be made here of the model of KRONIG and KORRINGA (1943), KRONIG (1949), introduced in a treatment of stopping by a free gas. A special feature of their liquid picture of the system is the appearance of a viscosity, η, leading to the dielectric constant (2.15). KRONIG and KORRINGA applied this semi-classical model not only for distant collisions, where it does not differ from (2.6) or (2.14), but even for close collisions where the viscosity governs the motion. One gets a formula similar to (4.7), though with $8\,\eta/3\,\varrho$ instead of $\hbar$ in the logarithm. It is hardly desirable in this way to replace quantum theory by classical viscosity. Still, KRONIG and KORRINGA found the screening by polarization at large distances prior to KRAMERS.

We now turn to the treatment of a degenerate Fermi gas, where (3.5), (3.7), and (3.8) apply. Of course, these more involved formulae should be regarded merely as approximate estimates of the behaviour of the gas. The reason for their application here is that we wish to see whether deviations from (4.7) or

* It is to be noted that we have proved here the formula (4.7) only for such dilute gases where the Maxwell distribution applies (cf. § 5).

(4.7) will occur. For instance, one might find a change by a factor inside the logarithm in (4.7), which would be difficult to discover in the treatments of, e. g., KRAMERS (1947) and PINES (1953).

Introduce first the equation (3.5) in (4.3), so that the energy loss is expressed by the functions f_1 and f_2, integrated over the variables u and z. The logarithmic term, L, in (4.5) takes the form

$$L = \frac{6}{\pi}\int_0^{v/v_0} u\,du \int_0^{\infty} z^3\,dz\, \frac{f_2(u,z)}{(z^2+\chi^2 f_1(u,z))^2+(\chi^2 f_2(u,z))^2}, \qquad (4.8)$$

where $\chi^2 = e^2/(\pi\hbar v_0)$. The quantity χ^2 varies only slowly with the density of electrons, and is for metallic electrons somewhat less than unity. The density of the gas enters only through χ^2 and through the upper limit, v/v_0, in the integration over u. The functions f_1 and f_2 are given by (3.7) and (3.8), respectively. The value of L in (4.8) depends most directly on f_2, i. e. the imaginary part of ε, while f_1 is important only for small values of z, which implies that the polarization is of significance for distant collisions.

For the case of velocities v high compared with v_0 one may proceed as follows in evaluating L. Divide the integration in two parts, $u < u^1$ and $u > u^1$, where u^1 is a constant somewhat larger than unity. The integration over $u < u^1$ will give some constant. For $u > u^1$ there are two contributions, one from the region where f_2 is finite, or $|u-z| < 1$, corresponding to close collisions. The other contribution arises from the resonance at longer distances, where $z^2 + \chi^2 f_1(u, z) = 0$. Using the formulae (3.7), (3.8), and (3.10), we find that the two last mentioned contributions are equal to $\log(v/v_0 u^1)$, and adding the result for $u < u^1$ we have

$$L = \log\left\{C(\chi)\,\frac{2mv^2}{\hbar\omega_0}\right\}, \qquad (4.9)$$

where the quantity C is expected to depend on the value of χ. Of course, for small densities, and accordingly thermal velocities of the electrons in the gas, the value of C will tend to unity and equation (4.7) results. In the opposite extreme of high densities, i.e. $\chi^2 \ll 1$, a simple numerical computation indicates that C

again approaches unity. Intermediate densities, $\chi^2 \sim 1$, give values of C not much less than 1. In most cases one will then not be far in error in assuming $C = 1$.

For low velocities of the particle, by which we understand values of v small compared to the maximum velocity v_0 of the electrons in the gas, there is a considerable difference between (4.8) and the simpler picture resulting in (4.6). We can here in the limit replace u by 0 in f_1, and f_2, in the denominator in (4.8), and using (4.5) it is seen that

$$\left.\begin{aligned} \frac{dE}{dR} &= \frac{4 z_1^2 e^4 m^2}{3 \pi \hbar^3} v \cdot C_1(\chi) \\ C_1(\chi) &= \int_0^1 \frac{z^3 \, dz}{(z^2 + \chi^2 f_1(0, z))^2}, \end{aligned}\right\} \tag{4.10}$$

showing that for small v the energy loss is proportional to the velocity. The case of low velocities was treated on semi-classical lines in a paper by FERMI and TELLER (1947). The formula obtained by them results from (4.10) if in C_1 the function $f_1(0, z)$ is put equal to $f_1(0, 0) = 1$, and at the same time $\chi^2 \ll 1$, which gives $C_1 = -\log \chi - \frac{1}{2} \approx \frac{1}{2} \log (v_0 \hbar / e^2)$.

One may replace (4.10) by a simpler and yet approximately correct formula. For this purpose we notice that the function $C_1(\chi)$ over a wide range of densities increases nearly as $\varrho^{1/4}$. It can further be useful to compute numerically some values of the expression (4.8) in the interval $0 < (v/v_0) < 1$, where generally it is not allowed to put $u = 0$ in f_1 and f_2. We find that, instead of (4.10), the following logarithmic term in (4.5) can be used

$$L = \text{const} \cdot \left(\frac{2 m v^2}{\hbar \omega_0} \right)^{\frac{3}{2}}, \tag{4.11}$$

where the constant is of the order 0.1—0.05. The formula should be applicable in the density interval $0.02 \lesssim \chi^2 < 1$ but, because of the difference from the result for electrons at rest, (4.11) is less reliable than (4.9). According to (4.7) and (4.11), the function L is both for high and low velocities a function of the argument $2 m v^2 / \hbar \omega_0$, at least in the first approximation. This result can be of convenience in the handling of more involved problems (LINDHARD and SCHARFF, 1953).

We have supposed above that it is allowed to use a perturbation treatment, even for quite low velocities of the particle. However, when v is sufficiently low—of the order $z_1 e^2/\hbar$—there enters the new feature of capture and loss of electrons, and the particle will carry electrons which to some extent screen the field around it. For this and other reasons it is assumed, usually, that a perturbation treatment is applicable only when $v > z_1 e^2/\hbar$ (cf., e.g., BETHE and LIVINGSTON, 1937). Nevertheless, the present type of perturbation method is not much in error even for lower values of v. In connection with this point it is of importance to notice, first, that in a treatment of the present kind there is actually a screening of the field and an inclusion of capture and loss in so far as such effects can be contained in linear field equations. Moreover, the crucial entity in the collisions is the relative velocity, which for low v is given by the electron velocities in the system, and not by the much smaller quantity v. These remarks are of course not limited to a free gas of electrons, but apply for atomic systems in general.

If one regards the stopping problem from a frame of reference moving with the particle, the formulae (4.10) and (4.11) give the loss of momentum per unit time of an electron gas streaming, with uniform velocity v, past a point charge $z_1 e$. One may therefore make a comparison between the stopping of a slow particle and calculations of residual resistance, and since in the latter case the rate of momentum loss is proportional to the electric current according to Ohm's law, the proportionality to v in (4.10) is not accidental. More quantitatively, if in this formula one allows a reduction to the semi-classical approximation, i.e. $f_1(0, z) = f_1(0, 0)$, there is complete equivalence to a familiar formula for the resistance as caused by foreign scattering centres in a metal.* The approximate empirical justification of the resistance formula, both as regards proportionality to $z_1^2 \cdot v$ and as to absolute values, will again show that for low v it is not unjustified to apply a perturbation approach in the stopping problem.

In the considerations in the previous sections, the damping of the electronic motion by resistance was introduced in a most

* Cf., for instance, MOTT and JONES: Properties of Metals and Alloys, Oxford University Press 1936, p. 294. The approximation used there is of a similar kind as that of FERMI and TELLER (1947).

cursory manner. The above remarks on one aspect of the resistance problem will give an indication as to how the resistance results from the dielectric constant in a linear field equation.

Straggling.

The straggling in energy loss provides an example of the direct application of equation (1.14). We consider, as before, a heavy particle moving through the system. It is then possible to find the probability of absorption of quanta $\hbar\omega$, by use of (1.14), (1.4), (4.1), and (4.2).

We shall not enter on details of the problem and only compute the straggling for the case where the particle has penetrated a distance through the system sufficient to ensure that the distribution around the average is nearly Gaussian. For this case the standard deviation, Ω, in the distribution in energy loss is determined by the equation

$$\Omega^2 = \langle (dE)^2 \rangle - \langle dE \rangle^2 = \int_0^\infty d\omega\, p(\omega)\, \hbar^2 \omega^2, \tag{4.12}$$

where $p(\omega)\, d\omega$ is the differential probability for energy transfer $\hbar\omega$. The formula (4.12) gives the fluctuation for the limit where Bose-Einstein statistics reduces to Boltzmann statistics, i. e. when field quanta are only rarely excited. Therefore, equation (4.3) corresponds to the straggling

$$\Omega^2 = -\, dR\, \frac{z_1^2\, e^2}{\pi v^2}\, \mathrm{Im} \left\{ \int_0^\infty \frac{dk}{k} \int_0^{kv} 2\, d\omega\, \frac{\omega^2}{\varepsilon^l(k, \omega)} \right\}. \tag{4.13}$$

As before, we consider first the case where the electrons in the gas are initially at rest. One then introduces (3.4) in (4.13); we shall quote the resulting expression only in two limiting cases. For high velocities is obtained the well-known formula

$$\Omega^2 = 4\, \pi\, z_1^2 e^4\, \varrho\, dR, \tag{4.14}$$

where relativistic corrections are omitted. Usually, the relativistic correction is a multiplication by the factor $(1 - v^2/2c^2)/(1 - v^2/c^2)$, cf. (3.21). It is often simpler, instead of the absolute value of

the straggling, to give the relative value Ω^2/dE. When (4.14) holds we find $\Omega^2/dE = mv^2/L$. In the extreme of low velocities of the particle, which, however, according to (4.6) must obey $mv^2/\hbar\omega_0 > 1$, the relative straggling can easily be shown to approach a minimum value

$$\frac{\Omega^2}{dE} = 2^{\frac{1}{2}}\,\hbar\omega_0, \tag{4.15}$$

determined by $\hbar\omega_0$, the lowest possible energy transfer to the gas.

For a degenerate Fermi gas one gets of course the result (4.14) for high velocities v. For low velocities we quote the formula corresponding to (4.10). We introduce (3.5) in (4.13) and obtain

$$\left.\begin{aligned} \Omega^2 &= 4\,\pi\, z_1^2\, e^4\, \varrho\, dR \cdot L_\Omega, \\ L_\Omega &= \frac{12}{\pi}\left(\frac{v_0}{v}\right)^2 \int_0^{v/v_0} u^2\, du \int_0^\infty \frac{z^4\, dz\, f_2(u,z)}{(z^2+\chi^2 f_1(u,z))^2+(\chi^2 f_2(u,z))^2}. \end{aligned}\right\} \tag{4.16}$$

Next, u is put equal to zero in f_1 and f_2 in the denominator. On account of (3.8), L_Ω then takes the simpler form

$$L_\Omega = \frac{3}{2}\left(\frac{v}{v_0}\right)^2 \int_0^1 \frac{z^4\, dz}{(z^2+\chi^2 f_1(0,z))^2}, \tag{4.17}$$

showing that for low velocities the straggling, Ω^2, behaves as v^2. As in (4.10) the integral in (4.17) may be approximated by a simple function. Over a wide region of densities it is found that L_Ω is nearly proportional to $\varrho^{-1/2}$, and dividing by (4.11) the following estimate of the relative straggling is obtained for low velocities

$$\frac{\Omega^2}{dE} \simeq (5\, mv^2 \cdot \hbar\omega_0)^{\frac{1}{2}}. \tag{4.18}$$

As was to be expected, the individual energy transfers can thus be interpreted as the velocity v of the particle times an effective momentum of the electrons.

Multiple scattering.

A problem similar to straggling in energy loss is the multiple scattering of the particle. It is well known that the major contribution to the multiple scattering arises from collisions with the atomic nuclei of the substance, because they can have charges high compared with the electron charge. Still, let us briefly resume the results for the multiple scattering contributed by the gas of electrons. The mean of the square of the total angle of deflection of the particle, ψ^2, can be found by summing the squares of the individual momentum transfers perpendicular to the path, and dividing by the momentum squared of the particle. The general formula for the multiple scattering is then

$$\psi^2 = -\frac{z_1^2 e^2 \hbar\, dR}{\pi M^2 v^4} \operatorname{Im} \left\{ \int_0^\infty \frac{dk}{k} \int_0^{kv} 2\, d\omega \, \frac{k^2 - \dfrac{\omega^2}{v^2}}{\varepsilon^l(k,\omega)} \right\}, \tag{4.19}$$

if all angles entering in the formula are small. We apply here the present expressions for the dielectric constant, where the effect of the atomic nuclei is omitted. When evaluating (4.19) for high velocity of the particle, using either (3.4) or (3.5), we find that ψ^2 behaves similarly as the energy loss, but the distant collisions, i. e. the resonance collisions, will now be suppressed. One shows easily that

$$\psi^2 = \frac{m}{M} \cdot \frac{dE}{2E'} \tag{4.20}$$

where $E = Mv^2/2$ is the energy of the particle. In the limit of low velocities the dielectric constant (3.5) — (3.8) may be used. With the same approximation as in (4.17) it is seen that here

$$\psi^2 = \frac{\Omega^2}{4E^2}. \tag{4.21}$$

Width of states.

The width of the particle states, Γ, we introduce as $\hbar$ times the previously mentioned transition probability per unit time,

$$\Gamma = -\frac{2\,z_1^2\,e^2}{\pi v}\,Im\left\{\int_0^\infty \frac{dk}{k}\int_0^{kv} d\omega\,\frac{1}{\varepsilon^l(k,\omega)}\right\}, \tag{4.22}$$

and a similar expression results for the transverse field. It is interesting that the width is a classical property, in the sense that (4.22) does not depend directly on $\hbar$.

For high velocities we get, from (3.4) and (4.22),

$$\Gamma = \frac{z_1^2\,e^2}{2\,v}\,\omega_0 \log\frac{4\,mv^2}{\hbar\omega_0}. \tag{4.23}$$

For low velocities, in a Fermi gas, (3.5) leads to

$$\Gamma = \frac{z_1^2\,e^4\,m}{\pi\hbar^2}\cdot\frac{v}{v_0}\cdot C_\Gamma(\chi), \quad C_\Gamma = \int_0^1 \frac{z^2\,dz}{(z^2+\chi^2 f_1(0,z))^2} \sim \frac{\pi}{4\,\chi}. \tag{4.24}$$

Self-energy.

The self-energy of a charged particle is closely connected with its energy loss. The self-energy may be found from the energy density of the field in equation (1.11); in order to obtain the self-energy due to the medium we must subtract the self-energy in vacuum. In so far as the recoil of the source can be neglected a calculation of the self-energy is straightforward, since the fields are immediately given. We consider the longitudinal contribution, which alternatively may be found as one half of the potential at the position of the particle. Introducing the source density (4.1), as corresponding to a point charge with velocity v, we perform a calculation similar to that resulting in (4.3) and obtain the following expression for the self-energy

$$u(v) = \frac{z_1^2\,e^2}{2\,\pi v}\int_0^\infty \frac{dk}{k}\int_{-kv}^{+kv} d\omega\left\{\frac{1}{\varepsilon^l(k,\omega)} - 1\right\}, \tag{4.25}$$

where the mass M of the particle should be large.

The expression (4.25) is quite similar to that describing the energy loss, (4.3), or the transition probability to other states. We may, in fact, combine the two in one formula for the complex self-energy, U,

$$U = u - i\frac{\Gamma}{2} = \frac{z_1^2 e^2}{2\pi v}\int_0^\infty \frac{dk}{k}\int_{-kv}^{+kv} d\omega \left\{\frac{1}{\varepsilon_C^l(k,\omega)} - 1\right\}, \qquad (4.26)$$

where $\Gamma/\hbar$ is the previously deduced transition probability. Further, $\varepsilon_C^l(k, \omega)$ represents $\varepsilon^l(k, \omega)$ for $\omega > 0$, and $\varepsilon^{l*}(k, \omega)$ for $\omega < 0$. We note that the transverse analogue of (4.26) may be derived directly from (4.4).

For high velocities we get from (4.25) and (3.4) the simple result

$$u = -z_1^2 \frac{\pi e^2}{4\hbar v} \cdot \hbar\omega_0, \qquad (4.27)$$

leading to quite small values when v is large. The contributions to (4.27) arise from polarization at distances about equal to the adiabatic impact parameter, v/ω_0, and not from the very small probability of carrying an electron. Similarly, if the particle is at rest, or moving slowly, we can for a degenerate gas employ the dielectric constant (3.5), (3.6) and obtain

$$u = -\frac{2 z_1^2 me^4}{\pi^2 \hbar^2}\int_0^\infty dz \cdot \frac{f_1(0, z)}{z^2 + \chi^2 f_1(0, z)}, \qquad (4.28)$$

where $f_1(u, z)$ is given by (3.7) and the integral, accordingly, is about equal to $\pi/2\chi$. Thus, we have found an approximate expression for the binding of, e. g., a proton in the gas. In the semi-classical limit one finds for the self-energy, to second power in the velocity of the particle,

$$u = \frac{z_1^2 me^4}{\pi\hbar^2} \cdot \frac{1}{\chi}\left\{-1 + \frac{1}{6}\left(1 - \frac{\pi^2}{16}\right)\frac{v^2}{v_0^2}\right\}. \qquad (4.29)$$

One may attempt to improve eq. (4.26) by including the recoil of the particle. Since in (4.26) the frequency ω is given by $\hbar\omega = \hbar k \cdot v = \Delta p \cdot (\partial E/\partial p)$, it would seem natural in quantum theory simply to replace $\hbar\omega$ by $\hbar k \cdot p/m + \hbar^2 k^2/2m$. The proper formula is, however, slightly more complicated. In the Appendix is derived the term to be added to the free particle Hamiltonian in the wave equation (cf. (A. 12)).

§ 5. Discussion of the approximation method.

In the calculations of the dielectric constants in § 2 and § 3, to first order in e^2, it was assumed that the electrons may be regarded as freely moving particles. This assumption is a central one in the Sommerfeld theory as well as in the Thomas-Fermi model. We shall discuss in how far the first order field equation, starting from free particles, is a good approximation to the correct field equations. Our treatment of this question is based on the pictorial semi-classical ideas. We consider two characteristic quantities; first, the self-energy as implying a momentum-energy relation different from that of a free particle, and second, the damping of motion of a single particle, giving rise to an uncertainty in its energy. The smallness of both damping and self-energy is a sufficient condition for the use of the first order approximation.

But let us first mention one simple argument showing a peculiar self-contained feature of the first order treatment. We notice that the semi-classical first order equations in § 2 depend only on the charge density $e\varrho$, on the ratio e/m, and on the particle velocities $\vec{v}_i = \vec{p}_i/m$. The distribution of particles of finite charge can thus be replaced by a continuum distribution having the same mass and charge density and the same velocity distribution, with the result that the first order equations remain unchanged. However, since the interaction of a charge with itself is at least proportional to the charge squared, all self-energies can be neglected in the continuum description, where charges may be considered infinitesimal, and the first order equations of § 2 become classically exact. This consideration makes apparent a most important feature of the field equations calculated: the first order perturbation treatment can be accurate, and at the same time the resulting induced field is by no means a small quantity.

Returning to an actual gas of particles, let us first explain what we mean by a self-consistent treatment in the semi-classical approximation. In § 2 we deduced the dielectric constants, the approximation being that the wavelengths of the electrons were very small compared to those of the field. On this assumption all contributions were found to arise from electrons at the surface

of the distribution in momentum space, with velocity v_0 and momentum p_0. We made a distinction between mv_0 and p_0, although they were equal in calculations to first order in e^2. In a more accurate classical treatment the only change is that mv differs from p. We now assume that, for a given electron density, the electrons occupy all states below the momentum $p_0 = (3\pi^2\varrho)^{1/3}\cdot\hbar$, with none above. This gives in any case a well-defined approximation to classical self-consistent equations. Since the contributions arise only from the surface of the distribution, our description involves merely one new quantity

$$\alpha = \frac{mv_0}{p_0} = \frac{m}{p_0}\frac{\partial E}{\partial p}\bigg|_{p=p_0}, \tag{5.1}$$

E being the energy of the electron.

This description can be applied when excitations involve states in the neighbourhood of the Fermi surface. Therefore, it can be used even in the quantum mechanical calculation in § 3, but only for fields of long wavelength and low frequency. In fact, if we define an effective electron mass as $m^* = p_0/v_0 = m/\alpha$, we merely need – in the formulae in § 2 and § 3 – replace throughout m by m^*; this should be done in ω_0^2 and χ^2, too. It is seen that then we obtain, e. g., the well-known dependence of spin paramagnetism and orbital diamagnetism (pp. 23 and 34) on effective electron mass. Further, while m^* refers to slowly varying fields, the opposite case of very large wave vectors and frequencies is accurately described by the uncorrected dielectric constants in § 3.

The energy E is the sum of the kinetic energy $mv^2/2$ and the self-energy u, given by (4.25). In order to bring out the essential features we now simplify the description. We can write approximately, for not too high velocities, $u(v) = u_0 + u_2\cdot v^2/2v_0^2$, as in eq. (4.29). It is then found from (5.1) and (4.29) that α is determined by

$$\frac{1}{\alpha} = 1 + C\cdot\chi^3, \tag{5.2}$$

where $C = (\pi/3)\cdot(1 - \pi^2/16)$. The previously introduced quantity $\chi^2 = me^2/\pi\hbar p_0$ is thus a measure of the applicability of the free electron picture and the first order field equations.† The depend-

† In LINDHARD (1946), eq. (25), a somewhat different consideration led to $\chi^2 < 1$ as the condition for the validity of the first order semi-classical picture.

ence on χ^3 in (5.2) is not due to the simplifying assumptions made as regards the self-energy, but holds accurately in the semi-classical picture as long as α is close to unity; a numerical computation shows that C is very nearly 0.5. Clearly, α is close to unity in dense gases, and for densities corresponding to electrons in metals α is slightly smaller than 1. For gases of low densities the approximations made are apparently not justified.

For a dilute gas in temperature equilibrium, we may similarly ask for the limit of applicability of the free electron picture, and of the Maxwell velocity distribution. Since, in the description of such gases, Planck's constant does not enter, the parameter characterizing the gas is, instead of χ^2, a quantity proportional to

$$\chi_1^2 = \frac{e^2 \varrho^{\frac{1}{3}}}{\theta}, \tag{5.3}$$

which just corresponds to the ratio between the binding energy and the temperature energy. It may here be recalled that the field equations for a Boltzmann gas were found roughly to correspond to those of a degenerate gas, only v_0 being replaced by the temperature velocity of the particles. On the same lines as above we then find an equation for α similar to (5.2), but with χ_1^2 instead of χ^2. Thus, the simple Maxwell distribution is only valid for χ_1^2 small compared to unity.

Mention may be made of the curious case of a one-dimensional degenerate gas, where the first order semi-classical dielectric constant was found to be $\varepsilon = 1 - \omega_0^2/(\omega^2 - v_0^2 k^2)$. The contributions to ε arise from particles at the surface of the distribution in momentum space, and the field surrounding such particles moving with constant velocity v_0 has non-zero Fourier components only for $\omega = \pm v_0 k$. For this self-field ε is infinite, so that the self-energy and its derivative with respect to momentum vanishes. Thus, we have proved that the first order treatment is semi-classically exact, in agreement with the result of TOMONAGA (1950).

So far the ratio α and the momenta were considered, for simplicity, as real quantities. However, we found earlier a damping of motion, by no means negligible compared to the self-energy. Let us regard the resulting effect in the case of χ^2 small, so that the free particle picture is approximately valid.

The imaginary part, $\Gamma/2$, of the energy is then given semi-classically by eq. (4.26). If again we use the estimate asymptotically valid for low velocities, we get according to (4.24) proportionality between Γ and the velocity of the particle; the corresponding complex momentum has an imaginary part equal to the constant $(\pi/8)\cdot(me^2/\hbar)\chi$. Accordingly, α is approximately given by

$$\alpha = 1 - C\cdot\chi^3 + i\frac{\pi^2}{8}\chi^3. \tag{5.4}$$

In the present limit of $\alpha \simeq 1$ we could of course easily get estimates more accurate than (5.4). In this connection we note that Γ, while rising at first linearly with the velocity, will reach a maximum for energies somewhat above the top of the Fermi distribution. For still higher energies Γ decreases slowly towards zero. The value of the imaginary part of (5.4) is only an approximate estimate, due to the simplifying assumptions made; but the dependence on χ^3 has general validity in the semi-classical approximation.

For a finite imaginary part of α some interesting consequences result in the formulae, as (2.4), for the dielectric constants, where then v_0 is to be replaced by $\alpha p_0/m = \alpha\hbar\,(3\pi^2\varrho)^{1/3}/m$.† For instance, for long wavelengths, where a phenomenon similar to mesons is observed (cf. p. 25), the inclusion of the imaginary part of (5.4) implies a width of the states of the field quanta, corresponding to a life-time for the field quanta at rest

$$\frac{1}{\tau'} = \frac{\omega_0}{2}\,\mathrm{Im}\,(\alpha) \approx \frac{1}{8\cdot 3^{\frac{1}{2}}}\cdot\frac{me^4}{\hbar^3}, \tag{5.5}$$

independent of the gas density for small χ's.

It is, perhaps, not without interest to make here a qualitative comparison with atomic nuclei. Indeed, in spite of the differences between electromagnetic and mesonic couplings, the

† The resulting improved dielectric constants are of the causal type, where the imaginary part does not change sign with ω. The causal dielectric constants are particularly convenient in higher order treatments.

By introducing in the transverse dielectric constant the effect of the self-energy, as expressed by α, we observe that the term $i\,(3\,\pi\omega_0^2/4\,\omega k v_0)\cdot(mv_0/p_0)$ in (2.7) remains unaffected. This result is important in the so-called anomalous skin effect, which is governed just by the mentioned term (cf. II).

general features of the results obtained above should also apply for a system of nucleons. One will expect the appearance of complex self-energies of the nucleons, of similar type as found above. The semi-empirical proposal by FESHBACH et al. (1953) may be regarded as a simplified description of such kind, although it will be evident that both the real and imaginary parts of the self-energy must vary with particle excitation.

The considerations in this paragraph and in § 4, aiming at improved electromagnetic field equations, at the same time give improved equations for the particle field. The latter equations are also of the type of Maxwell's equations in matter, in being of high order, and involving polarization and absorption effects. The equations for the electromagnetic field are important for the dynamics of the system, e. g. in the case of interaction with external charges. The equations for the particle field are useful for instance when a particle, identical with those in the gas, enters the system from the outside. In general, the particle equations are of importance when one is able to discern approximately the motion of one, or a few, individual particles.

Summary.

The paper treats the behaviour of a gas of charged particles, preferably a degenerate gas. It is pointed out that the dynamic properties of this system are contained in equations for the electromagnetic field merely, of type of Maxwell's equations in matter. By the field is meant, classically, the field induced by and acting on external, classical charges. A systematic treatment on this basis implies great simplifications in the theory.

The interpretation of general field equations is discussed, and the manner in which they account for absorption processes. The dielectric constants, defining linear field equations, are computed in a number of cases, to first and higher order in e^2, using both classical and quantum description of the particle motion. As a demonstration of the method is treated energy dissipation by a charged particle moving through the system, and its self-energy. Further is discussed self-consistent field equations, and the improved electronic equations of motion.

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Appendix.

Improved particle field equations.

In § 4 was deduced, classically, the self-energy and damping of motion of a charged particle moving in a field obeying the equation

$$\Delta \varepsilon \Phi = -4\pi \varrho_0, \tag{A. 1}$$

where the operator ε is the longitudinal dielectric constant. We shall now derive the corresponding quantum effect, to first order in e^2. In this way one gets an improved equation for the particle field, of similar type as the equation for the electromagnetic field, (A. 1). In the derivation we employ a technique more advanced than that used for the deduction of the equation (A. 1) in § 3. Still, the calculation will not be performed in the least cumbersome manner, but is hoped to be the more illustrative as regards the effects involved.

The field equation for the particle is taken to be the non-relativistic one,

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta\right)\psi = e\Phi\psi, \tag{A. 2}$$

and at the same time we introduce $\varrho_0 = e\,\psi^*\psi$ in (A. 1). The equations (A. 1), (A. 2) are conveniently written on integral form

$$\psi(x) = \psi^{in}(x) - e\int S^R(x-x')\,\Phi(x')\,\psi(x')\,dx', \qquad \text{(A. 3)}$$

$$\Phi(x) = \Phi^{in}(x) + e\int D^R(x-x')\,\psi^*(x')\,\psi(x')\,dx', \qquad \text{(A. 4)}$$

where x stands for (x, y, z, t). Moreover, $S^R(x-x')$ is the retarded solution of (A. 2) when the right side of the equation is replaced by $-\delta(x-x')$; $D^R(x-x')$ is the analogous solution of (A. 1) with a source $-4\pi\delta(x-x')$ on the right. The incoming fields ψ^{in} and Φ^{in} are solutions of the uncoupled equations.

The coupled equations result in a scattering of the particle, which we calculate to second order from a series development in (A. 3) and (A. 4). For the outgoing field we find

$$\left.\begin{aligned}\psi^{out}(x) &= \psi^{in}(x) + e\int S(x-x')\,\Phi^{in}(x')\,\psi^{in}(x')\,dx' \\ &+ e^2\iint dx'\,dx''\,S(x-x')\cdot D^R(x'-x'')\,\psi^{*in}(x'')\,\psi^{in}(x'')\,\psi^{in}(x') \\ &- e^2\iint dx'\,dx''\,S(x-x')\cdot S^R(x'-x'')\,\Phi^{in}(x')\,\Phi^{in}(x'')\,\psi^{in}(x''),\end{aligned}\right\} \qquad \text{(A. 5)}$$

where $S = S^A - S^R$.

Consider the vacuum expectation value of $\psi^{*in}(x_1)\,\psi^{out}(x)$, which quantity can serve to characterize the scattering. The first order term in e is seen to vanish for symmetry reasons, and we get to second order

$$\left.\begin{aligned}\langle 0|\,\psi^{*in}(x_1)\,\psi^{out}(x)\,|0\rangle &= \langle 0|\,\psi^{*in}(x_1)\,\psi^{in}(x)\,|0\rangle \\ - e^2\iint dx'\,dx''\,S(x-x')\,\{\,&D^R(x'-x'')\langle 0|\,\psi^{*in}(x'')\,\psi^{in}(x')\,|0\rangle \\ + S^R(x'-x'')\langle 0|\,&\Phi^{in}(x')\,\Phi^{in}(x'')\,|0\rangle\,\}\,\langle 0|\,\psi^{*in}(x_1)\,\psi^{in}(x'')\,|0\rangle.\end{aligned}\right\} \qquad \text{(A. 6)}$$

In this equation we introduce

$$\langle 0|\,\psi^{*in}(x'')\,\psi^{in}(x')\,|0\rangle = -i\hbar\,S^-(x'-x''), \qquad \text{(A. 7)}$$

according to ordinary field quantization; the function S^- is the negative frequency part of S. As to the electromagnetic field the method of quantization of Peierls (1952), establishing the general connection with the Green's functions of the field equations, leads to

$$\langle 0|\,\Phi^{in}(x')\,\Phi^{in}(x'')\,|0\rangle = i\hbar\,D^+(x'-x''). \qquad \text{(A. 8)}$$

We wish to eliminate the coupling with the electromagnetic field and attempt to replace it by a new term in the particle field equation

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\Delta\right)\psi(x)+\int K(x-x')\,\psi(x')\,dx'=0. \qquad \text{(A. 9)}$$

The above scattering is obtained if, according to (A. 6), (A. 7), and (A. 8), K is of the form

$$K(x)=ie^2\hbar\,(-D^R(x)\cdot S^-(x)+S^R(x)\cdot D^+(x)). \qquad \text{(A. 10)}$$

It can be convenient to write this formula in terms of D^C and S^C, defined by $D^C=-2i\overline{D}+D^{(1)}$. To this purpose we note that the expression $-D^R(x)\,S^-(x)+S^R(x)\,D^+(x)$ is equal to

$$(-i/2)\left\{D^R(x)\,S^{(1)}(x)+D^{(1)}(x)\,S^R(x)\right\},$$

and a simple manipulation leads to

$$K(x)=\frac{i}{4}e^2\hbar\,S^C(x)\,D^C(x), \qquad \text{(A. 11)}$$

the equation holding only for the positive frequency part of the field, which is usually sufficient, since the incoming particle has positive energy. Substituting (A. 10) or (A. 11) for K in (A. 9) an improved equation for the particle field is obtained. For a Fourier component of the particle wave proportional to $\exp(i\vec{k}\cdot\vec{r}-i\omega t)$ the extra term in the field equation—which could be called a self-energy of the particle—is accordingly

$$\left.\begin{aligned}U(k,\omega)=-\frac{ie^2}{4\pi^3}\iint d^3k'\,d\omega'\,\frac{1}{\omega'+i\delta-\dfrac{\hbar k'^2}{2m}}\cdot\frac{1}{(\vec{k}-\vec{k}')^2}\\ \cdot\left(\frac{1}{\varepsilon_C(|\vec{k}-\vec{k}'|,\omega-\omega')}-1\right),\end{aligned}\right\} \qquad \text{(A. 12)}$$

where we have subtracted the vacuum term with $\varepsilon=1$; further $\varepsilon_C(k,\omega)=\varepsilon(k,|\omega|)$ and δ is a vanishing positive quantity.

In eq. (A. 12) we have of course in first approximation the connection $\omega = \hbar k^2/2\, m$ if the particle energy is approximately that of a freely moving particle.

When the mass m, is large, and for a given particle velocity $v = \hbar k/m$, the formula (A. 12) reduces to the semi-classical one, (4.26).

The above derivation of linear particle field equations from general linear electromagnetic field equations owes its feasibility and simplicity to the use of the quantization rule (A. 8). Evidently, one may give, on the same lines, a derivation of the electromagnetic field equations from general particle equations, and thus obtain mutually consistent equations for the two fields. In the text, this was performed, quantum mechanically, for free particles, but also for a more general case when only long waves or low frequencies were important.

References.

Alfvén (1950), Cosmical Electrodynamics, Oxford.
Bethe and Livingston (1937), Rev. Mod. Phys. **9**, 245.
Blatt and Weisskopf (1952), Theoretical Nuclear Physics, J.Wiley and Sons.
Bloch (1933), ZS. f. Phys. **81**, 363.
Bloch (1934), Helv. Phys. Acta **7**, 385.
Bohm and Gross (1949), Phys. Rev. **75**, 1851, 1864.
Bohm and Pines (1951), Phys. Rev. **82**, 625.
Bohm and Pines (1952), Phys. Rev. **85**, 338.
Bohm and Pines (1953), Phys. Rev. **92**, 609.
Bohr, A., (1948), Dan. Mat. Fys. Medd. **24**, No. 19.
Bohr, N. (1913), Phil. Mag. (6) **25**, 10.
Bohr, N. (1948), Dan. Mat. Fys. Medd. **18**, No. 8.
Fermi (1924), ZS. f. Phys. **29**, 315.
Fermi (1940), Phys. Rev. **57**, 485.
Fermi and Teller (1947), Phys. Rev. **72**, 399.
Feshbach, Porter and Weisskopf (1953), Phys. Rev. **90**, 166.
Frank and Tamm (1937), C. R. Acad. Sci. URSS. **14**, 109.
Goldberger (1948), Phys. Rev. **74**, 1269.
Heisenberg (1947), Zeits. f. Naturf. **2a**, 185.
Klein (1945), Arkiv f. mat., astr. o. fys. **31A**, No. 12.
Kramers (1947), Physica **13**, 401.
Kronig and Korringa (1943), Physica **10**, 406.
Kronig (1949), Physica **15**, 667.
Landsberg (1949), Proc. Phys. Soc. A **162**, 806.
Lindhard (1946), Arkiv f. mat., astr. o. fys. **33A**, No. 4.
Lindhard and Scharff (1953), Dan. Mat. Fys. Medd. **27**, No. 15.
Lindhard (1953), Phil. Mag. (7) **44**, 916.
London (1950), Superfluids, Vol. 1, New York.
Peierls (1952), Proc. Roy. Soc. Lond. *A*, **214**, 143.
Pines (1952), Phys. Rev. **85**, 931.
Pines (1953), Phys. Rev. **92**, 626.
Ruthemann (1948), Ann. d. Physik (6) **2**, 113.
Tomonaga (1950), Progr. Theor. Phys. **5**, 544.
Weizsäcker, v., (1933), Ann. d. Physik (5) **17**, 869.
Wohlfarth (1950., Phil. Mag. (7) **41**, 534.

Added in proof: The reader is further referred to an interesting manuscript by T. Kinoshita and Y. Nambu (Institute for Advanced Study, Princeton, N. J.), The Collective Description of Many-Particle Systems.

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